ENGINEERING DYNAMICS

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VOLUME IV

INTERNAL-COMBUSTION ENGINES

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21 NOV 1956

BLACKIE & SON LIMITED
LONDON AND GLASGOW
1954

PREFACE

Textbooks as well as elementary lecture-courses covering the syllabus of Engineering Mechanics have to attach primary importance to simplified, rather academic, problems. For want of space and time, the more difficult problems met with in engineering practice are either omitted or cannot be treated as fully as would be required in actual practice. For this reason, most Technical Colleges have instituted special courses in Higher Mechanics dealing with the more complicated problems encountered by the practical designer or research engineer.

The present book is intended to serve a purpose similar to that of these special lectures. It represents, so to say, a continuation of the usual textbooks, a book on " Engineering Mechanics for Post-graduates ". Throughout this book, it is therefore assumed that the reader knows his elementary mechanics and the mathematics that goes with it.

The title " Engineering Dynamics ", however, has been chosen not only in order to indicate that elementary engineering mechanics has been excluded but also in an endeavour to restore the original meaning of the word " Dynamics ". If according to KIRCHHOFF'S unsurpassed explanation " mechanics " is the science of motion and forces, then " kinematics " may be regarded as the science of motion and " dynamics " as the science of forces. " Dynamics " thus includes " statics ", dealing with the geometry of forces and the state of equilibrium, and " kinetics ", governing the effects of unbalanced forces and the relationship between force and motion. A vibration problem, for example, is a problem in dynamics and consists of a static part, viz. the determination of the restoring-force coefficients, and a kinetic part, viz. the calculation of natural frequencies or the shape of the vibration curve. If, as often happens in normal usage, the word " dynamics " is taken to be synonymous with " kinetics ", then the rather important static problems connected with vibrations would have to be excluded from consideration. Yet this static part very often constitutes the more difficult, if not always the more attractive, side of the problem.

The present book thus covers some of those static and kinetic problems which engineering practice has to solve by means of the scientific methods of mechanics. For the arrangement and treatment of the subject, the authors have always kept in mind that a technical problem is worth solving only if it can be applied in practice and that a solution is complete only when all the details of the problem can be numerically evaluated with a tolerable amount of arithmetical work. In spite of the importance the authors have attached to numerical results, they have always given prominence to the methods of solution, and a whole chapter of the book has been devoted entirely to a study of methods. Solutions are not given in outline merely, but every detail has been developed in the way in which

a designer must execute his calculations. The authors have also endeavoured to indicate for every problem the best and most complete solutions attainable by present-day methods.

A book of this kind is not meant to be read from cover to cover and the authors have therefore arranged that, apart from a few exceptions dictated by the subject-matter, nearly every one of the thirteen parts is self-contained.

Since the solutions are given in great detail, only a limited number of the most important technical problems in dynamics could be included in the book. The authors have selected these problems from objective and personal motives, thinking mainly (though not exclusively) of the needs of mechanical engineering and especially of engine design. Even then they have had to omit or shorten the treatment of those special subjects which have been extensively covered in existing publications, for example the governing of engines or the theory of curved shells. The authors feel they need hardly apologize for the fact that personal motives entered into their selection; they intended to deal mainly with those problems to which they could contribute something of their own. They hold that those can best grasp the inner meaning of a problem and explain it to others, who have themselves worked on it and contributed to its solution.

Throughout the book the subject-matter requires an intense co-operation from the reader. He may feel compensated by the thoroughly practical aim of the book which in many cases will guide him to final formulæ and tables for the calculation of technical problems. He may also find reward in the knowledge that the far-reaching methods of solution he learns to master by his collaboration will prove to be of very considerable value for the solution of further new problems.

The authors record their gratitude to several younger collaborators who have given their unstinting help in the writing and printing of the book. Messrs. P. Riekert, H. Ziegler, K. Schulz have critically revised the manuscript and the last-mentioned has also read all the proofs in conjunction with the authors. Messrs. A. Kimmel, E. Maier, H. Kauderer, and especially Mr. J. J. Koch have helped with many details. The publishers and their staff have always been very willing to discuss the—not always modest—wishes of the authors who take this occasion to record their special thanks to them.

The two authors finally wish to express that they carry common responsibility for all parts of this book which was inspired by the last-named author. May the book show how scientific work bridges frontiers of country and limits of language.

Delft and Stuttgart, August 1939.

C. B. Biezeno, R. Grammel.

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INTERNAL-COMBUSTION ENGINES

Part I

Inertia Forces

1. Introduction.—One of the most important problems associated with reciprocating engines, especially with high-speed internal-combustion engines, is that of balancing out the inertia forces. This involves two distinct steps: first, to calculate the reactive forces and moments at the bed-plate or engine support due to the moving masses; second, to discover how these forces and moments can be entirely or partly eliminated.

Usually, these forces and moments cannot be represented by explicit mathematical expressions; we have the choice of developing them either in power series of the argument $\lambda = r/l$ (crank radius/length of connecting-rod) or in Fourier series in terms of the crank angle ψ. In the following pages we shall use the second scheme, because with internal-combustion engines of compact design the value of λ can be as great as 0·4, resulting in unsatisfactory convergence of the power series (unlike the reciprocating steam engine), and also because the investigation of resonance is best handled with Fourier series.[1]

The problem of reduction or elimination of inertia reactions in internal-combustion engines is very different from that in multi-cylinder steam engines. In the latter, the crank offsets, the cylinder spacings, and even the piston weights, can be varied over wide ranges and can, therefore, be chosen for optimum inertia balancing. In internal-combustion engines generally—whether or not this is always true will not be debated here—more or less equal crank offsets, usually equal cylinder spacings, and always equal piston weights are specified, so that only a choice of firing order and the addition of counterweights remains for the balancing of inertia forces.

From the kinematics of the standard reciprocating mechanism (with no cylinder offset) we shall first obtain the Fourier series for the inertia forces and moments in it, applying these results to the one-cylinder engine and to the problem of balancing (§ 1), and shall then do the same thing for the multi-cylinder in-line engine (§ 2). Following this, we shall consider

[1] P. Carnac, "The Exponential Method in the Analysis of the Balance of Reciprocating Masses", *Engineering* 112 (1921), p. 778; P. Riekert, "Der Massenausgleich von Reihenmotoren", Dissertation, Stuttgart, 1928 (contains a very extensive bibliography).

engines with unequal crank offsets, unequal cylinder spacings, offset cylinders, and with counterweights (§ 3).

A second class of multi-cylinder engines is made up of V-, multi-row, and radial engines. First to be considered are engines with a single cylinder per row and with centrally attached connecting-rods (not articulated) (§ 4), similar engines with articulated connecting-rods (§ 5), and lastly, engines with several cylinders per row (§ 6).

The third class of engines comprises rotary and counter-rotating engines. We shall consider balancing for a perfectly regular (symmetric) design, the only important case (§ 7).

§ 1. The One-cylinder Engine

2. General remarks.—We shall consider first the standard reciprocating mechanism (not offset) shown in Fig. 1 and represented schematically in Fig. 2, where O, A, and B are the points where the axis of the main bearings, the crankpin axis, and the gudgeon-pin or cross-head axis intersect the

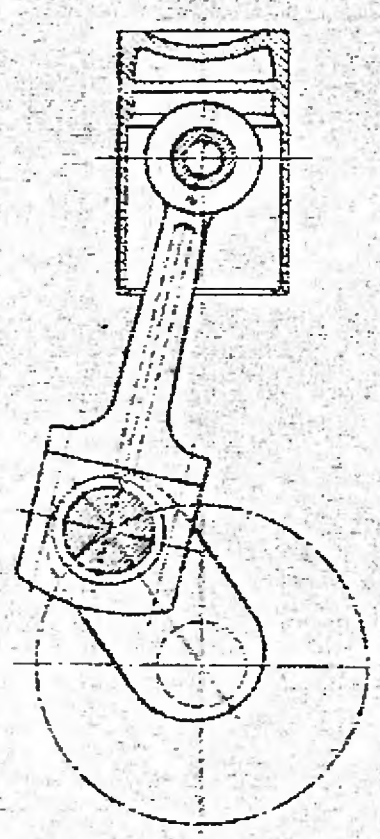


Fig. 1.—Crank mechanism of a Diesel engine.

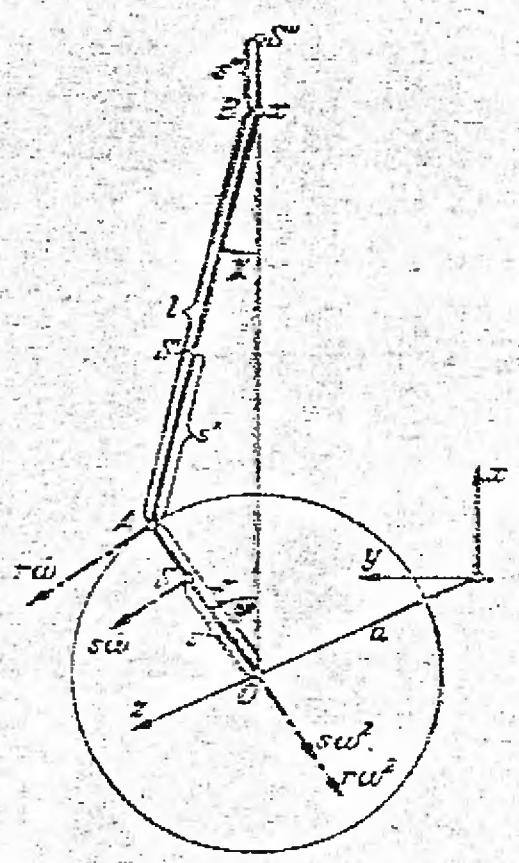


Fig. 2.—Schematic diagram of standard reciprocating mechanism.

plane of symmetry of the mechanism. The crank rotation ψ is measured from the upper dead-centre position. The total weight of the crank, crankpin, crank web, etc., is G and the corresponding centre of gravity is S. The throw of the crank OA is r, the distance OS is s, and $\omega = \dot{\psi}$, the angular velocity. G' is the weight of the connecting-rod, S' is its centre of gravity, l its length AB, and s' the distance AS'. G'' is the weight of the reciprocating

parts (piston, cross-head, and piston rod if any), S'' the centre of gravity and s'' the distance BS''. Finally, k is the radius of gyration of the crank about its axis of rotation and k' the radius of gyration of the connecting-rod about an axis through S' parallel to the crank axis.

Next, we establish a rectangular coordinate system whose z-axis coincides with the crank axis, positive being the direction moved by a right-hand screw turned through the angle $+\psi$ (Fig. 2). The (x, y)-plane is parallel to the symmetry plane of the mechanism, the positive x-direction being parallel to OB, while the positive y-direction corresponds to a right-hand system. We assume the middle plane of the mechanism to lie at a distance $z = a$ from the (x, y)-plane, and we call x the coordinate of B while χ is the angle turned by the connecting-rod from upper dead-centre BO.

We assume that S' lies on the line AB and that S, S', and S''' are in the middle plane of the mechanism. For convenience we introduce the ratio

$$\lambda = \frac{r}{l} \quad \left(\text{usually between } \frac{1}{6} \text{ and } \frac{2}{5}\right). \tag{1}$$

If the momentum—or the angular momentum with respect to the axis of coordinates—varies during the motion of the mechanism, the foundation must exert a force—or a moment—equal to the rate of change of momentum—or angular momentum. The equal and opposite reactions that the support receives from the mechanism are the sought-for inertia forces K or inertia moments M. The (x, y, z)-components of these are designated by X, Y, Z and M_x, M_y, M_z and are as follows:

X is the *longitudinal force* (in the line of stroke);
Y is the *transverse force* (normal to the stroke, in the plane of symmetry;
Z is zero;
M_x is the *yawing moment* about the x-axis;
M_y is the *pitching moment* about the y-axis;
M_z is the *rolling moment* about the z-axis.

In the calculation of these inertia forces and moments the internal forces and moments (explosion force, friction in the mechanism, etc.) have no part, since they consist always of equal and opposite pairs of forces and therefore cancel themselves. The word "support" must always be taken in the most general sense to mean *any* restraining connection between the mechanism and its surroundings. This includes not only the actual support itself but also the parts that transmit power from the engine, beyond the point up to which the weight G of the crank is calculated. For example, if the propeller of an aeroplane is included in the weight G of the engine parts, then the "support" includes the surrounding air which exerts forces and moments on the propeller[1].

[1] A. KIMMEL has investigated the proportion of M_z acting on the engine supports. See "Das Massendrehmoment des Flugmotors", *Luftfahrtforsch.* **20** (1943), p. 107.

3. The Inertia Forces and Moments.—In order to calculate the momentum of the mechanism and from it the inertia forces X and Y (as the negative rate of change of momentum) we may assume the mass of the crank G/g to be concentrated at its centre of gravity S, that of the reciprocating parts G''/g at their centre of gravity S'' and the mass of the connecting-rod G'/g divided between A and B, the ends of the rod, in such a way as to leave the centre of gravity S' unchanged. The corresponding weights are

$$G_1 = \left(1 - \frac{\varepsilon'}{l}\right) G' \text{ at } A \quad \text{and} \quad G_2 = \frac{\varepsilon'}{l} G' \text{ at } B. \tag{1}$$

Thus the equivalent system of four concentrated masses has the same centre of gravity as the mechanism itself and, consequently, has the same momentum and the same inertia forces.

The x- and y-components of the normal and tangential accelerations of the mass G/g are

$$\ddot{x} = -s\omega^2 \cos\varphi - s\dot{\omega} \sin\varphi \quad \text{and} \quad \ddot{y} = -s\omega^2 \sin\varphi + s\dot{\omega} \cos\varphi.$$

Similar expressions are obtained for the components of the accelerations ($r\omega^2$ and $r\dot{\omega}$) of the mass G_1/g. Using these, we obtain for the longitudinal and transverse inertia forces

$$X = \frac{G}{g}(s\omega^2 \cos\varphi + s\dot{\omega} \sin\varphi) + \left(1 - \frac{\varepsilon'}{l}\right)\frac{G'}{g}(r\omega^2 \cos\varphi + r\dot{\omega} \sin\varphi) - \left(\frac{\varepsilon'}{l}\frac{G'}{g} + \frac{G''}{g}\right)\ddot{x},$$

$$Y = \frac{G}{g}(s\omega^2 \sin\varphi - s\dot{\omega} \cos\varphi) + \left(1 - \frac{\varepsilon'}{l}\right)\frac{G'}{g}(r\omega^2 \sin\varphi - r\dot{\omega} \cos\varphi).$$

With the substitutions

$$Q = \frac{r}{g}\left(\frac{\varepsilon'}{l}G' + G''\right), \qquad Q' = \frac{1}{g}\left[sG + r\left(1 - \frac{\varepsilon'}{l}\right)G'\right] \tag{2}$$

these are simplified to

$$\left.\begin{aligned} X &= Q'(\omega^2 \cos\varphi + \dot{\omega} \sin\varphi) - Q\frac{\ddot{x}}{r}, \\ Y &= Q'(\omega^2 \sin\varphi - \dot{\omega} \cos\varphi). \end{aligned}\right\} \tag{3}$$

From the forces X and Y we get the inertia moments about the x- and y-axes immediately

$$M_x = -aY, \qquad M_y = aX. \tag{4}$$

In order to find the rolling moment M_z we must first calculate the angular momentum of the connecting-rod. Here, it is no longer possible to replace the rod by concentrated masses. Instead, G_1 and G_2 are supplied with moments of inertia $k_1^2 G_1/g$ and $k_2^2 G_2/g$ (with respect to the corresponding axes at A and B parallel to the crank axis) such that the moment of inertia of the whole rod about the axis at S' is unchanged. Then the angular momentum and the moment of inertia of the equivalent system are the same as

those of the actual connecting-rod. According to Huygens' theorem for moments of inertia about parallel axes,

$$k'^2G' = [k_1^2 + s'^2]G_1 + [k_2^2 + (l - s')^2]G_2$$

or

$$k_1^2G_1 + k_2^2G_2 = k'^2G' - s'^2G_1 - (l - s')^2G_2.$$

If equations (1) are substituted into the right-hand side of the preceding expression we get

$$k_1^2G_1 + k_2^2G_2 = [k'^2 - s'(l - s')]G'. \tag{5}$$

[It can be seen that if $k'^2 < s'(l - s')$ the equivalent masses must have negative moments of inertia. Although this is physically impossible, it can cause no difficulty here since only computation is involved and the principles of mechanics hold equally for positive and negative masses and moments of inertia.]

The angular momentum of the equivalent masses G_1/g and G_2/g consists of that due to the rotation $(-\dot\chi)$ of the connecting-rod $-\dot\chi(k_1^2G_1 + k_2^2G_2)/g$ plus the moments of the momentums of the masses G_1/g and G_2/g, namely, $\omega r^2G_1/g$ and zero (since the momentum of the mass G_2/g is directed along a line through O). Adding the angular momentum of the crank $\omega k^2G/g$ we get the total angular momentum

$$J = \frac{1}{g}[\omega(k^2G + r^2G_1) - \dot\chi(k_1^2G_1 + k_2^2G_2)].$$

Taking account of equation (5) and the first equation (1) and making the substitutions

$$R = \frac{\lambda}{g}[s'(l - s') - k'^2]G', \qquad R' = \frac{1}{g}\left[k^2G + r^2\left(1 - \frac{s'}{l}\right)G'\right] \tag{6}$$

we obtain for the rolling moment $M_z = -dJ/dt$, the expression

$$M_z = -R'\omega - R\frac{\ddot\chi}{\lambda}. \tag{7}$$

The magnitudes G, G', G'', r, l, s, s', k^2, and k'^2 appear only in the quantities Q, Q', R, and R'. We call Q and Q' the *reduced first moments*, and R and R' the *reduced moments of inertia* of the reciprocating mechanism.

Since it is intended to develop the inertia forces (3) and the inertia moments (4) and (7) in Fourier series in terms of the argument ψ it is necessary to express the quantities $\ddot x/r$ and $\ddot\chi/\lambda$ as functions of the crank angle ψ.

4. Kinematics of the Piston and Connecting-rod.—From Fig. 2

$$x = r\cos\psi + l\cos\chi, \qquad r\sin\psi = l\sin\chi. \tag{1}$$

After eliminating χ we find

$$\frac{x}{r} = \cos\psi + \frac{1}{\lambda}(1 - \lambda^2\sin^2\psi)^{1/2}. \tag{2}$$

If the right side of equation (2) is developed as a power series in λ there results

$$\frac{x}{r} = \frac{1}{\lambda} + \cos\psi - \frac{\lambda}{2}\sin^2\psi - \frac{\lambda^3}{8}\sin^4\psi - \frac{\lambda^5}{16}\sin^6\psi - \cdots. \tag{3}$$

However,

$$\left.\begin{aligned} \sin^2\psi &= \frac{1}{2}(1 - \cos 2\psi), \\ \sin^4\psi &= \frac{1}{8}(3 - 4\cos 2\psi + \cos 4\psi), \\ \sin^6\psi &= \frac{1}{32}(10 - 15\cos 2\psi + 6\cos 4\psi - \cos 6\psi). \end{aligned}\right\} \tag{4}$$

Substituting these expressions in (3) and rearranging, yields

$$\frac{x}{r} = A_0 + \cos\psi + \frac{1}{4}A_2\cos 2\psi - \frac{1}{16}A_4\cos 4\psi + \frac{1}{36}A_6\cos 6\psi - + \cdots, \tag{5}$$

together with the following power series for the coefficients:

$$\left.\begin{aligned} A_0 &= \frac{1}{\lambda} - \frac{1}{4}\lambda - \frac{3}{64}\lambda^3 - \frac{5}{256}\lambda^5 - \cdots, \\ A_2 &= \lambda + \frac{1}{4}\lambda^2 + \frac{15}{128}\lambda^5 + \cdots, \\ A_4 &= \frac{1}{4}\lambda^3 + \frac{3}{16}\lambda^5 + \cdots, \\ A_6 &= \frac{9}{128}\lambda^5 + \cdots. \end{aligned}\right\} \tag{6}$$

By differentiating twice with respect to time and setting $\dot{\psi} = \omega$ we get

$$\left.\begin{aligned} -\frac{\ddot{x}}{r} = \omega^2(&\cos\psi + A_2\cos 2\psi - A_4\cos 4\psi + A_6\cos 6\psi - + \cdots) + \\ &+ \omega\left(\sin\psi + \frac{1}{2}A_2\sin 2\psi - \frac{1}{4}A_4\sin 4\psi + \frac{1}{6}A_6\sin 6\psi - + \cdots\right). \end{aligned}\right\} \tag{7}$$

If we wish more terms of this equation we substitute in equation (2) the quantity

$$\frac{1}{\lambda}(1 - \lambda^2\sin^2\psi)^{1/2} = \sum_{k=0}^{\infty}(-1)^k\binom{\frac{1}{2}}{k}\lambda^{2k-1}\sin^{2k}\psi \tag{3a}$$

and replace equation (4) by the generalized expression

$$\sin^{2k}\psi = \frac{1}{2^{2k-1}}\sum_{h=0}^{k}(-1)^{k-h}\binom{2k}{h}\cos 2(k-h)\psi - \frac{1}{2^{2k}}\binom{2k}{k}. \tag{4a}$$

Then

$$\frac{x}{r} = A_0 - \cos\psi + \sum_{j=1}^{\infty}(-1)^{j-1}\frac{A_{2j}}{4j^2}\cos 2j\psi \tag{5a}$$

in which A_0 has no essential importance and

$$A_{2j} = 4j^2 \sum_{k=j}^{\infty} (-1)^{k-j} \binom{\frac{1}{2}}{k} \binom{2k}{k-j} \left(\frac{\lambda}{2}\right)^{2k-1} \qquad (j = 1, 2, \ldots). \tag{6a}$$

Finally

$$-\frac{\ddot{x}}{r} = \omega^2 \left(\cos\psi + \sum_{j=1}^{\infty} (-1)^{j-1} A_{2j} \cos 2j\psi\right) + \dot{\omega} \left(\sin\psi + \sum_{j=1}^{\infty} (-1)^{j-1} \frac{A_{2j}}{2j} \sin 2j\psi\right). \tag{7a}$$

The coefficients A_{2j} and $B_{2j} = A_{2j}/2j$ of equations (7) and (7a) can be easily evaluated for any crank/connecting-rod ratio λ by means of equations (6) or (6a) and are given in the following table. Approximately,

$$A_2 \approx \lambda, \qquad A_4 \approx \frac{1}{4}\lambda^3, \qquad A_6 \approx \frac{1}{14}\lambda^5. \tag{6b}$$

$\frac{1}{\lambda}\left(=\frac{l}{r}\right)$	A_2	A_4	A_6	$B_2 = \frac{1}{2}A_2$	$B_4 = \frac{1}{4}A_4$	$B_6 = \frac{1}{6}A_6$
2·5	0·4173	0·0182	0·0009	0·2087	0·0045	0·0001
3·0	0·3431	0·0161	0·0003	0·1715	0·0025	0·0001
3·5	0·2918	0·0062	0·0001	0·1459	0·0016	0·0000
4·0	0·2540	0·0041	0·0001	0·1270	0·0010	—
4·5	0·2250	0·0028	0·0000	0·1125	0·0007	—
5·0	0·2020	0·0021	—	0·1010	0·0005	—
5·5	0·1833	0·0015	—	0·0917	0·0004	—
6·0	0·1678	0·0012	—	0·0839	0·0003	—

Comparison of the more accurate tabulated values with those obtained from equation (6) shows that the difference in the A_2 remains less than 0·003 for terms up to and including λ^3, and less than 0·001 for terms including λ^5, when λ has its maximum expected value of 0·4. Moreover, the Fourier coefficient A_6 is at most 1/400 of A_2, and for the rather common crank/connecting-rod ratio $\lambda = 1:3{\cdot}5$, it is less than 1/1000 of A_2. Consequently, it is generally sufficient to use Fourier terms up to $\cos 4\psi$ and $\sin 4\psi$ and to go to $\cos 6\psi$ and $\sin 6\psi$ only when there is particular danger of resonance.

Next, we consider the calculation of χ. The second equation (1) can be written in the form

$$\cos\chi = (1 - \lambda^2 \sin^2\psi)^{1/2} = 1 - \frac{\lambda^2}{2}\sin^2\psi - \frac{\lambda^4}{8}\sin^4\psi - \frac{\lambda^6}{16}\sin^6\psi - \cdots.$$

When this is differentiated with respect to time and the result divided by $\sin\chi = \lambda\sin\psi$ there results

$$\dot{\chi} = \omega\left(\lambda\cos\psi + \frac{1}{2}\lambda^3\sin^2\psi\cos\psi + \frac{3}{8}\lambda^5\sin^4\psi\cos\psi + \cdots\right). \tag{8}$$

Introducing equations (4) and expressing the product $\cos 2k\psi \cos\psi$ as $\frac{1}{2}\cos(2k+1)\psi + \frac{1}{2}\cos(2k-1)\psi$ yields

$$\chi = \omega\lambda\left(C_1\cos\psi - \frac{1}{3}C_3\cos 3\psi + \frac{1}{5}C_5\cos 5\psi - + \cdots\right) \tag{9}$$

with the coefficients

$$\left.\begin{aligned} C_1 &= 1 + \frac{1}{8}\lambda^2 + \frac{3}{64}\lambda^4 + \cdots,\\ C_3 &= \frac{3}{8}\lambda^2 + \frac{27}{128}\lambda^4 + \cdots,\\ C_5 &= \frac{15}{128}\lambda^4 + \cdots. \end{aligned}\right\} \tag{10}$$

Differentiating once more with respect to time gives

$$\left.\begin{aligned} \frac{\ddot{\chi}}{\lambda} = &-\omega^2(C_1\sin\psi - C_3\sin 3\psi + C_5\sin 5\psi - + \cdots) +\\ &+\omega\left(C_1\cos\psi - \frac{1}{3}C_3\cos 3\psi + \frac{1}{5}C_5\cos 5\psi - + \cdots\right). \end{aligned}\right\} \tag{11}$$

More generally

$$\chi = \omega\lambda\sum_{j=1}^{\infty}(-1)^{j-1}\frac{C_{2j-1}}{2j-1}\cos(2j-1)\psi \tag{9a}$$

with

$$C_{2j-1} = (2j-1)^2\sum_{k=j-1}^{\infty}(-1)^{k+1}\frac{2k-1}{k+j}\binom{\frac{1}{2}}{k}\binom{2k}{k-j+1}\left(\frac{\lambda}{2}\right)^{2k} \quad (j = 1, 2, \ldots). \tag{10a}$$

where $\binom{0}{0} = 1$.

Finally

$$\frac{\ddot{\chi}}{\lambda} = -\omega^2\sum_{j=1}^{\infty}(-1)^{j-1}C_{2j-1}\sin(2j-1)\psi + \omega\sum_{j=1}^{\infty}(-1)^{j-1}\frac{C_{2j-1}}{2j-1}\cos(2j-1)\psi. \tag{11a}$$

The coefficients C_{2j-1} and $D_{2j-1} = C_{2j-1}/(2j-1)$ appearing in (11) and (11a) are given in the following table for the most important values of the crank/connecting-rod ratio λ. Note that, approximately,

$$C_1 \approx 1, \qquad C_3 \approx \frac{3}{8}\lambda^2, \qquad C_5 \approx \frac{1}{8}\lambda^4. \tag{10b}$$

$\frac{1}{\lambda}\left(=\frac{r}{l}\right)$	C_1	C_3	C_5	$D_1 = C_1$	$D_3 = \frac{1}{3}C_3$	$D_5 = \frac{1}{5}C_5$
2·5	1·021	0·066	0·004	1·021	0·022	0·001
3·0	1·014	0·044	0·002	1·014	0·015	0·000
3·5	1·010	0·032	0·001	1·010	0·011	—
4·0	1·008	0·024	0·001	1·008	0·008	—
4·5	1·006	0·019	0·000	1·006	0·006	—
5·0	1·005	0·015	—	1·005	0·005	—
5·5	1·004	0·013	—	1·004	0·004	—
6·0	1·003	0·011	—	1·003	0·004	—

Since $C_7 = 0{\cdot}00018$ and $D_7 = C_7/7 = 0{\cdot}00003$ for the extreme case of $\lambda = 0{\cdot}4$, the Fourier terms containing $\cos 7\psi$ and $\sin 7\psi$ can be entirely neglected. Similarly, the terms with $\cos 5\psi$ and $\sin 5\psi$ amount to only a few thousandths of those with $\cos\psi$ and $\sin\psi$ and may be ignored except in cases where resonance is especially dangerous.

5. Fourier Series for the Inertia Forces and Inertia Moments.—In order to complete the development of the longitudinal and transverse forces in Fourier series it is only necessary to insert the value of $-\ddot{x}/r$ from (4, 7) or (4, 7a) in the expressions for X and Y:

$$\left.\begin{aligned} X = {} & \omega^2[(Q+Q')\cos\psi + Q(A_2\cos 2\psi - A_4\cos 4\psi + A_6\cos 6\psi - + \cdots)] + \\ & + \dot{\omega}[(Q+Q')\sin\psi + Q(\tfrac{1}{2}A_2\sin 2\psi - \tfrac{1}{4}A_4\sin 4\psi + \tfrac{1}{6}A_6\sin 6\psi - + \cdots)]. \end{aligned}\right\} \quad (1)$$

$$Y = \omega^2 Q'\sin\psi - \dot{\omega}Q'\cos\psi. \quad (2)$$

From these we get the yawing moment $M_z = -aY$ and the pitching moment $M_y = aX$ directly. In the same way the rolling moment is found from (3, 7) and (4, 11) or (4, 11a).

$$\left.\begin{aligned} M_x = {} & \omega^2 R(C_1\sin\psi - C_3\sin 3\psi + C_5\sin 5\psi - + \cdots) - \\ & - \dot{\omega}P' - \dot{\omega}R(C_1\cos\psi - \tfrac{1}{3}C_3\cos 3\psi + \tfrac{1}{5}C_5\cos 5\psi - + \cdots). \end{aligned}\right\} \quad (3)$$

The Fourier coefficients of the terms $\cos j\psi$ and $\sin j\psi$ appearing in the expressions for X, Y, M_x, M_y and M_z are designated, respectively, as *forces and moments of the jth order*. Those containing the factor ω^2 are *steady* and those with $\dot{\omega}$ are *unsteady* forces or moments. Since in normal engines the angular acceleration $\dot{\omega}$ is, at most, a few per cent of ω^2, the unsteady forces and moments are of secondary importance in comparison with the steady forces and moments.

In the standard reciprocating mechanism without offset the following inertia forces and moments are found:

(1) A transverse force and a yawing moment (about the x-axis), both of the first order.

(2) A longitudinal force and a pitching moment (about the y-axis) of the first and of all even orders.

(3) A rolling moment (about the crank axis) of zero and of all uneven orders.

By examining equations (1) to (3) it can be seen that the unsteady terms (except for $-\dot{\omega}P'$) can be obtained from the steady terms by replacing in the latter the quantities

$$\left.\begin{aligned} & \omega^2,\ \cos\psi,\ \sin\psi,\ \cos j\psi,\ \sin j\psi \ [\text{and later } \cos(j\psi+\beta) \text{ and } \sin(j\psi+\beta)] \text{ by} \\ & \dot{\omega},\ \sin\psi,\ -\cos\psi,\ \frac{1}{j}\sin j\psi,\ -\frac{1}{j}\cos j\psi \ \left[\text{and later } \frac{1}{j}\sin(j\psi+\beta) \text{ and } -\frac{1}{j}\cos(j\psi+\beta)\right]. \end{aligned}\right\} \quad (4)$$

Since all the engines that will be discussed later (except those in § 5) are made up entirely of standard reciprocating mechanisms we can take

advantage of this fact to simplify our presentation. In the following pages we shall give the steady terms in full (those with ω^2) and then write the symbol $((\omega))$ to indicate that if the substitutions indicated in (4) are made in the previous expressions containing ω^2 the unsteady terms will result. In this abbreviated form equations (1) to (3) are written

$$X = \omega^2[(Q + Q')\cos\psi + Q(A_2\cos 2\psi - A_4\cos 4\psi + - \cdots)] + ((\omega)), \tag{1}$$

$$Y = \omega^2 Q' \sin\psi + ((\omega)), \tag{2}$$

$$M_z = \omega^2 R(C_1\sin\psi - C_3\sin 3\psi + - \cdots) + ((\omega)) - \omega K. \tag{3}$$

By this method of writing the equations we indicate that in general we are concerned only with the steady state, but that the unsteady terms are obtainable if required. Since modern internal-combustion engines run very uniformly, we may replace ω^2 by the quantity $\pi^2 n^2/900$ in which n is the number of revolutions per minute. Since the steady terms increase with the square of the speed they assume very large values in high-speed engines.

6. **Inertia Balancing of the One-cylinder Engine.**—We now seek to discover to what extent inertia balancing can be achieved in a single standard reciprocating mechanism.[1] It follows immediately from (5, 2) that the transverse force Y vanishes for all values of the crank rotation ψ when and only when the reduced first moment Q' equals zero. According to (3, 2), for the one-cylinder engine this condition is

$$sG + r\left(1 - \frac{s'}{l}\right)G' = 0. \tag{1}$$

Since r, l, G, and G' are naturally positive, the quantities s and $(1 - s'/l)$ cannot both be positive. This requirement can be met either by lengthening the connecting-rod beyond the cross-head (so that $s' > l$) or by extending the crank web backwards past the main bearing pin (so that $s < 0$). The first alternative is practically never used because of construction difficulties, but the second has long been in use in the form of counterweights attached to both crank webs (Fig. 3) or to symmetrically placed flywheels. According to equation (1) these counterweights must be such that the whole crank resting in its main bearings with axis horizontal and loaded with the fraction $(1 - s'/l)G'$ of the connecting-rod weight G' is in static equilibrium for any angle of rotation.

Next, it follows from (5, 1) that the longitudinal force X vanishes when and only when both the reduced first moment Q equals zero and the sum $Q + Q'$ equals zero. Therefore, according to (3, 2) we have as the conditions

[1] R. Mollier, "Der Beschleunigungsdruck der Schubstange", *Zeit. VDI* 47 (1903), p. 1639; H. Lorenz, "Massenwirkungen von Getriebegruppen", *Zeit. VDI* 62 (1918), p. 562; R. Bethenods, "Massenausgleich bei Kurbeltrieben, insbesondere durch Gegengewichte", *Zeit. VDI* 64 (1920), p. 42; M. Tolle, "Regelung der Kraftmaschinen", 256 pp., 3rd. ed. Berlin, 1921.

for balancing of longitudinal forces

$$s'G' + lG'' = 0 \quad \text{and} \quad sG + r(G' + G'') = 0. \tag{2}$$

Since r, l, G, G', and G'' are naturally positive, these equations can be satisfied only by negative values of s' and s. The second condition is easily satisfied by counterweights on symmetrically placed flywheels (less easily by counterweights on the cranks). However, the first condition involves serious construction difficulties: the connecting-rod would have to be extended beyond the crankpin, and the extension furnished with a supplementary mass such

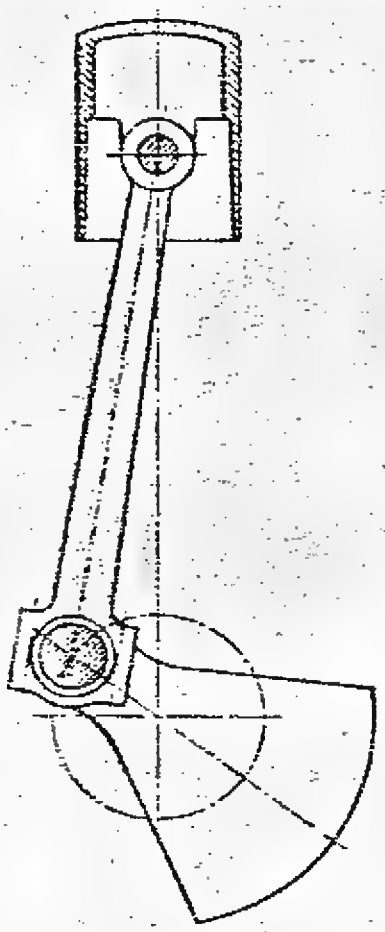


Fig. 3.—Standard reciprocating mechanism with counterweighted crank.

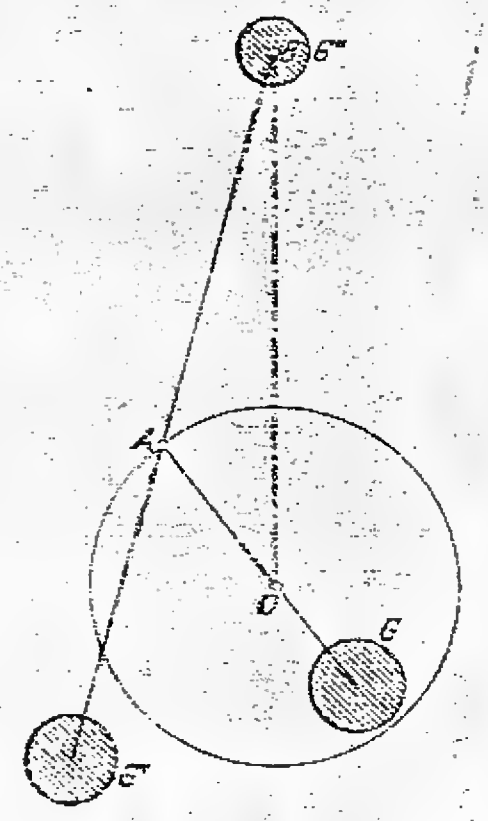


Fig. 4.—Standard reciprocating mechanism balanced for transverse and longitudinal forces, and for yawing and pitching moments.

that the connecting-rod, thought of as pivoted at the crankpin, together with the weight G'' of the reciprocating masses (concentrated at B) is in static equilibrium in any position. This state of balance is shown in Fig. 4 in which the cross-hatched circles represent the weights of the crank, connecting-rod, and reciprocating masses, all considered concentrated at their respective centres of gravity. The system G', G'' must have its centre of gravity at the centre of the crankpin A, and the whole system G, G', G'' must have its centre of gravity at the crank axis O, according to the second equation (2). On account of the relatively great weight of the reciprocating masses G'', the counterweights on the extension of the connecting-rod and on the cranks would have to be extremely large. In large engines there are always almost insuperable difficulties to this.

Consequently, since the condition $Q = 0$ is unobtainable for practical reasons, we must content ourselves with the condition $Q + Q' = 0$ (the second equation (2)). Then, according to (5, 1) there is no longer complete longitudinal-force balancing, but only balancing of the first-order forces. Thus, the smaller the ratio λ the better is the state of balance. However, we must not forget that with $Q = 0$, the condition $Q + Q' = 0$ for approximate longitudinal-force balancing prevents satisfying the condition $Q' = 0$ for exact transverse-force balancing. In practice, we compromise, replacing the two conditions $Q' = 0$ and $Q + Q' = 0$ by the single condition $\varkappa Q + Q' = 0$, in which $\varkappa$ lies between 0 and 1. Setting $\varkappa = 0$ gives complete transverse balancing, and making it equal to unity results in approximate longitudinal balancing. We select $\varkappa$ nearer to unity or nearer to zero depending on whether reduction of longitudinal or of transverse forces is preferred. This preference will depend on the means of supporting the engine. The condition for approximate longitudinal-transverse balancing becomes

$$sG + r\left\{\left[1 - (1-\varkappa)\frac{s'}{l}\right]G' + \varkappa G''\right\} = 0 \qquad (0 \leqq \varkappa \leqq 1). \tag{3}$$

This condition is satisfied by introducing counterweights on the backward extensions of the crank webs or on symmetrically placed flywheels. If $\varkappa = \frac{1}{2}$ (recommended for normal use) the counterweights are the averages of those required by (1) and by (2).

The remaining uncompensated transverse force consists of a steady component and an unsteady component (of the first order). With the substitution

$$G^* = \frac{s'}{l}G' + G'' \tag{4}$$

these become

$$Y_1 = \omega^2 Q' = -\varkappa r\omega^2 \frac{G^*}{g}, \qquad Y_1' = -\omega Q' = \varkappa r\omega \frac{G^*}{g}. \tag{5}$$

Similarly, the uncompensated longitudinal force has steady and unsteady first-order components

$$X_1 = \omega^2(Q + Q') = (1-\varkappa) r\omega^2 \frac{G^*}{g}, \qquad X_1' = \omega(Q + Q') = (1-\varkappa) r\omega \frac{G^*}{g}. \tag{6}$$

The higher-order components are

$$\left.\begin{aligned} X_{2j} &= (-1)^{j-1}\omega^2 Q A_{2j} = (-1)^{j-1} r\omega^2 A_{2j}\frac{G^*}{g}, \\ X_{2j}' &= (-1)^{j-1}\omega Q \frac{A_{2j}}{2j} = (-1)^{j-1} r\omega \frac{A_{2j}}{2j}\frac{G^*}{g}, \end{aligned}\right\} \qquad (j = 1, 2, 3, \ldots). \tag{7}$$

It appears that the first-order longitudinal force decreases with increasing $\varkappa$, and for any value of $\varkappa$ between zero and unity it is smaller than it would be in the absence of counterweights. The longitudinal forces of higher order are independent of $\varkappa$ and cannot be reduced in this way. (One

method by which at least one longitudinal-force component of higher order can be eliminated will be described in Sect. 15.) The transverse force is zero for $\varkappa = 0$ and increases with increasing $\varkappa$. It should be noted that if $\varkappa$ is made large enough this force becomes greater than it would have been with no counterweights. This is normally true for $\varkappa = 1$ since

$$\varkappa G^* \equiv \varkappa\left(\frac{s'}{l}G' + G''\right) \geq gQ' \equiv sG + \varkappa\left(1 - \frac{s}{l}\right)G'.$$

It must be remembered that by the word "force" we have meant the maximum value of the actual force which pulsates with the rotation ψ of the crank at the rate ψ or $2j\psi$. The actual uncompensated steady force of the first order consists of a vector $\mathbf{K}_1$ with components $X_1 \cos\psi$ and $Y_1 \sin\psi$ ((5) and (6)). This vector rotates oppositely to the crank. Its end describes an ellipse with the semi-axes X_1 and $|Y_1|$ (Fig. 5). Likewise, the uncompensated unsteady force of the first order is a rotating vector $\mathbf{K}_1'$ which traces out an ellipse with semi-axes X_1' and Y_1'. The situation is similar when the forces are entirely uncompensated. In that case the forces X_1, Y_1, X_1', and Y_1' are not given by (5) and (6) but are obtained from (5, 1) and (5, 2). The forces of higher order pulsate in the x-direction and have no other component.

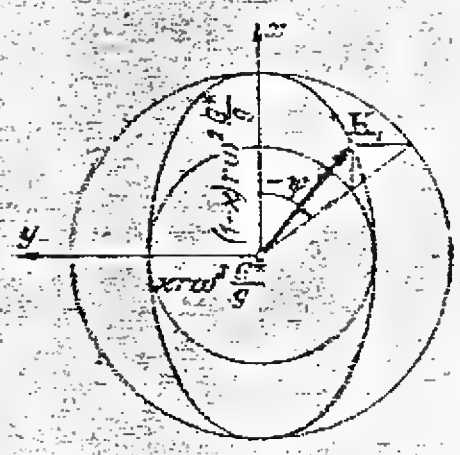


Fig. 5.—Uncompensated steady force of the first order.

The yawing and pitching moments decrease with decreasing transverse and longitudinal forces (2, 4). Balancing the rolling moment M_x (5, 3) requires that the reduced moments of inertia R and R' (3, 6) be zero, or

$$k^2 = s'(l - s') \quad \text{and} \quad k^2 G + \varkappa^2\left(1 - \frac{s}{l}\right)G' = 0. \tag{8}$$

We recognize that these two conditions are mutually incompatible. Since G and G' are positive quantities the second condition requires that $s' > l$. However, with the first condition this is impossible. Consequently, complete rolling-moment balance in the one-cylinder engine cannot be achieved. (However, see Sect. 15.)

Fortunately, the terms containing ω are relatively unimportant. Thus, if the unsteady part of M_x is ignored, balancing of the steady component involves only the condition $R = 0$, or

$$k^2 = s'(l - s'). \tag{9}$$

In practice this is a sufficient condition for rolling-moment balance. It can also be expressed in the following two forms:

$$\frac{k^2 - s^2}{s'} = l \quad \text{or} \quad \frac{k^2 - l - s'^2}{l - s'} = s. \tag{9a}$$

These state that the equivalent length of the connecting-rod (the length of a simple pendulum of the same period) must be l for suspension either at the crankpin bearing or at the cross-head (gudgeon-pin) bearing.

If we examine existing connecting-rods we find that this condition (9) or (9a) is apparently too little known and is usually not exactly satisfied, although there are scarcely any construction difficulties involved. Satisfying this condition will, of course, give smoother engine operation. If the con-

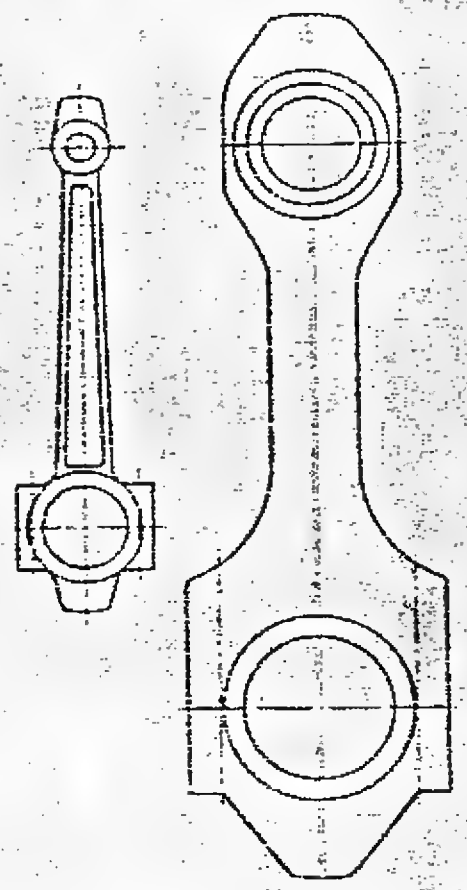


Fig. 6.—Connecting-rod with optimum rolling-moment balancing.

necting-rod consists of a cylindrical or uniformly tapered body with a symmetrical bearing at each end, then $k''^2 < s'(l - s')$ always. It is easily seen that for the body alone $k''^2 < s'(l - s')$, while for the two ends alone, assumed concentrated at the centres of the bearings, $k''^2 = s'(l - s')$. In order to make $k''^2 = s'(l - s')$ for the whole rod it is only necessary to extend the rod by a small amount beyond one or both bearings. Fig. 6 shows two such connecting-rods designed for optimum rolling-moment balancing.

§ 2. The In-line Engine

7. The Two-cylinder Engine.—(*a*) *Two-stroke cycle.*—In the single-acting two-cylinder two-stroke engine the two cranks are generally offset by the angle $2 \times \pi/2 = 180°$. If we take the forces and moments for the one-cylinder engine (Sect. 5) and add to them forces and moments for equal

values of Q, Q', R, and R' but with $\psi + 180°$ instead of ψ, we get

$$\left.\begin{aligned} X &= 2\omega^2 Q(A_2 \cos 2\psi - A_4 \cos 4\psi + A_6 \cos 6\psi - + \cdots) + ((\omega)), \\ Y &= 0. \end{aligned}\right\} \quad (1)$$

$$\left.\begin{aligned} M_x &= -\omega^2 a Q' \sin\psi - ((\omega)), \\ M_y &= \omega^2 a(Q + Q') \cos\psi + ((\omega)). \end{aligned}\right\} \quad (2)$$

$$M_z = -2\omega R'. \quad (3)$$

The origin of coordinates is taken on the crank axis midway between the cylinders which are a distance a apart (Fig. 7).

The moments M_x and M_y are calculated with respect to this origin; this will be the reference point for moments and moment balancing from now on. This means that when moments are in balance for axes through this point there need not necessarily be moment balance with respect to another axis. (This can be the case when the forces are not in balance.) In general, to get the moments for a particular set of axes it will be necessary to add to the moments M_x and M_y the moments of the forces X and Y. For an unsymmetrically supported engine this consideration may be important.

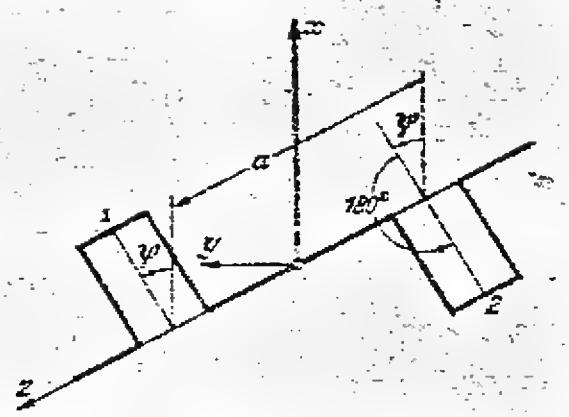


Fig. 7.—Crankshaft of two-cylinder two-cycle engine.

According to (1) and (3) the longitudinal force of first order, the entire transverse force, the higher orders of the yawing and pitching moments as well as the steady part of the rolling moment are automatically compensated. The longitudinal forces of higher order and the unsteady part of the rolling moment are not necessarily negligible. (However, see Sect. 15, p. 38.) For the yawing and pitching moments of the first order the same balancing conditions hold as for the transverse force and the longitudinal force of first order in the one-cylinder engine; optimum balancing is obtained by use of counterweights as in (6, 3). For cranks with intermediate bearings, i.e. with separated inner crank webs, these counterweights (which do not affect the longitudinal force X) are placed at both inner and outer crank webs and at equal distances from the cylinder axes. However, other counterweight arrangements may be equivalent to this so far as moment balancing is concerned, provided the centrifugal forces for any crank position, e.g. for $\psi = 0$, are equal to a couple of the same value as that of the counterweights calculated according to (6, 3). One takes advantage of this fact in crankshafts without intermediate bearings and whose two inner crank webs form a single oblique piece. If the inner crank webs are a single piece perpendicular to the crank axis and if the crankshaft is symmetric, the counterweights on the inner webs can be omitted since they would only pull equally against each other.

The uncompensated forces of higher order pulsate in the x-direction. Both the steady and the unsteady components of the yawing and pitching moments can again be represented by vectors which trace out ellipses in the (x, y)-plane.

The double-acting two-cylinder two-stroke engine is discussed in Sect. 12.

(b) *Four-stroke cycle.*—The single-acting two-cylinder four-stroke engine is easily handled. Since equal firing intervals require the cranks to be offset by the angle $4 \times \pi/2 = 360°$ or $0°$, the conditions for inertia balancing are exactly those given in Sect. 6 for the one-cylinder engine, with doubled values of G, G', and G''. The yawing and pitching moments with respect to axes intersecting on the crankshaft midway between cylinders vanish on account of symmetry.

The double-acting two-cylinder four-stroke engine has its cranks offset by 180° and is exactly like the single-acting two-stroke engine with respect to balancing.

8. The Three-cylinder Engine.—In the balancing of the three-cylinder engine it is unnecessary to distinguish between two-stroke and four-stroke types. For the two-stroke engine the normal crank-offset angle is $2\pi/3 = 120°$, and for the four-stroke $4\pi/3 = 240°$. These angles are equivalent to each other and result in similar (or mirror image) crank arrangements. If we add to the forces and moments for a single cylinder given in Sect. 5 those caused by identical quantities Q, Q', R, and R' and with $\psi + 120°$ and $\psi + 240°$ instead of ψ, there results

$$\left.\begin{aligned} X &= 3\omega^2 Q(A_6 \cos 6\psi - A_{12} \cos 12\psi + - \cdots) + ((\omega)), \\ Y &= 0. \end{aligned}\right\} \tag{1}$$

$$\left.\begin{aligned} M_x &= -\sqrt{3}\,\omega^2 a Q' \cos\psi - ((\omega)), \\ M_y &= -\sqrt{3}\,\omega^2 a[(Q + Q')\sin\psi - Q(A_2 \sin 2\psi + A_4 \sin 4\psi - A_8 \sin 8\psi - \\ &\qquad - A_{10} \sin 10\psi + \cdots)] - ((\omega)), \end{aligned}\right\} \tag{2}$$

$$M_z = -3\omega^2 R(C_3 \sin 3\psi - C_9 \sin 9\psi + C_{15} \sin 15\psi - + \cdots) - ((\omega)) - 3\omega R'. \tag{3}$$

Here, a is the distance between two adjacent cylinders; the origin is taken at the crank axis in line with the centre cylinder, whose crank rotation is ψ (Fig. 8).

It can be seen that the whole transverse force vanishes, as do the longitudinal forces up through the fifth order. The remaining longitudinal forces of sixth and higher orders divisible by six cannot be ignored without further investigation (Sect. 15). However, on account of their smallness ($A_{2\nu}$ from the table in Sect. 4) they are generally unimportant. For the first-order yawing and pitching moments the same balancing conditions apply as for the transverse and longitudinal forces of first order in the one-cylinder engine, namely, compensation by use of counterweights according to (6, 3), or by any other arrangement of weights to give equivalent centrifugal forces.

On crankshafts without intermediate bearings and in which inner crank webs of adjacent cylinders are continuous and perpendicular to the crank axis, each pair of inner counterweights may be replaced by a single weight statically equivalent to the pair with respect to centrifugal forces. Of course, there still remains a pitching moment proportional to Q, of the second, fourth, eighth, and all even orders not divisible by six. In addition to the unbalanceable unsteady term of zero order, the rolling moment contains terms of third, ninth, and successive odd orders divisible by three. These latter can be easily eliminated by making $R = 0$, as in (6, 9).

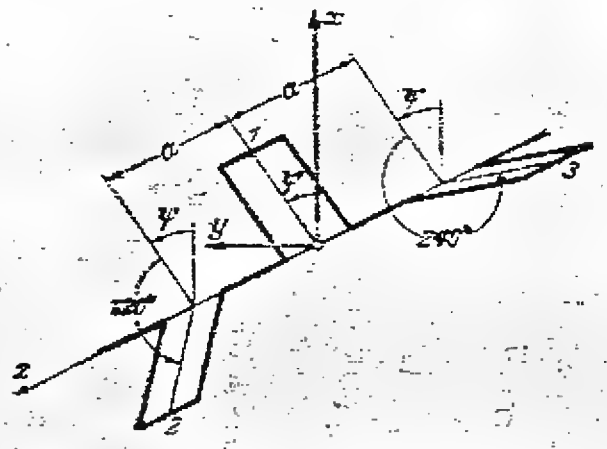


Fig. 8.—Crankshaft of a three-cylinder engine.

9. **Four-cylinder Engine.**—(a) *Two-stroke cycle.*—In the four-cylinder two-stroke engine the cranks of cylinders that fire consecutively are normally offset $2\pi/4 = 90°$. From the equations in Sect. 5 it can be shown that for this type of engine

$$\left.\begin{aligned} X &= -4\omega^2 Q(A_4 \cos 4\varphi + A_8 \cos 8\varphi + \cdots) - (\{\omega\}), \\ Y &= 0, \end{aligned}\right\} \tag{1}$$

$$M_x = -4\omega R'. \tag{2}$$

The whole transverse force vanishes, as do the components of the longitudinal force through the third order. The remaining components, of the fourth and higher orders divisible by four, cannot be eliminated practically (however, see Sect. 15) but are very small. Of the rolling moment there remains only the unsteady component of zero order.

The yawing and pitching moments depend on the arrangement of the engine. Fig. 9 shows three types of cranks for the four-cylinder two-stroke engine. At the left of the figure they are sketched in perspective; in the centre are the crank diagrams; on the right the cylinder-order diagrams. In the crank diagram the cranks are shown projected on the (x, y)-plane as seen from the positive z-direction. In the cylinder-order diagram the four cylinders are represented side by side. Two sets of numbers are shown: the Arabic numbers 1 2 3 4 give the angular order of the cranks, the Roman numerals I II III IV the relative positions of the cylinders. The latter system is normally used in industry; the former (taken off the crank diagram) is the more suitable for computation. The three-crank arrangements shown in Fig. 9 would be designated as follows:

(*a*) 1 2 3 4 or I II III IV; (*b*) 1 2 4 3 or I II IV III;

(*c*) 1 3 2 4 or I III II IV.

The first set of numbers is read off the cylinder-order diagram, moving to the right from 1; the second set is taken from the crank diagram going from I counter-clockwise.

Since no constraint is involved in assigning crank 1 to cylinder I the number of possible crank arrangements equals the number of permutations of the numbers 2, 3, 4, namely, six. These six are actually three pairs of mirror-image arrangements (reflected in the (x, z)-plane or by exchanging the numbers 2 and 4), i.e. 1 2 3 4 or 1 4 3 2, 1 2 4 3 or 1 4 2 3 and 1 3 2 4 or 1 3 4 2, the members of each pair being dynamically equivalent. Thus the arrangements (a), (b), (c) above are all that need be considered here.

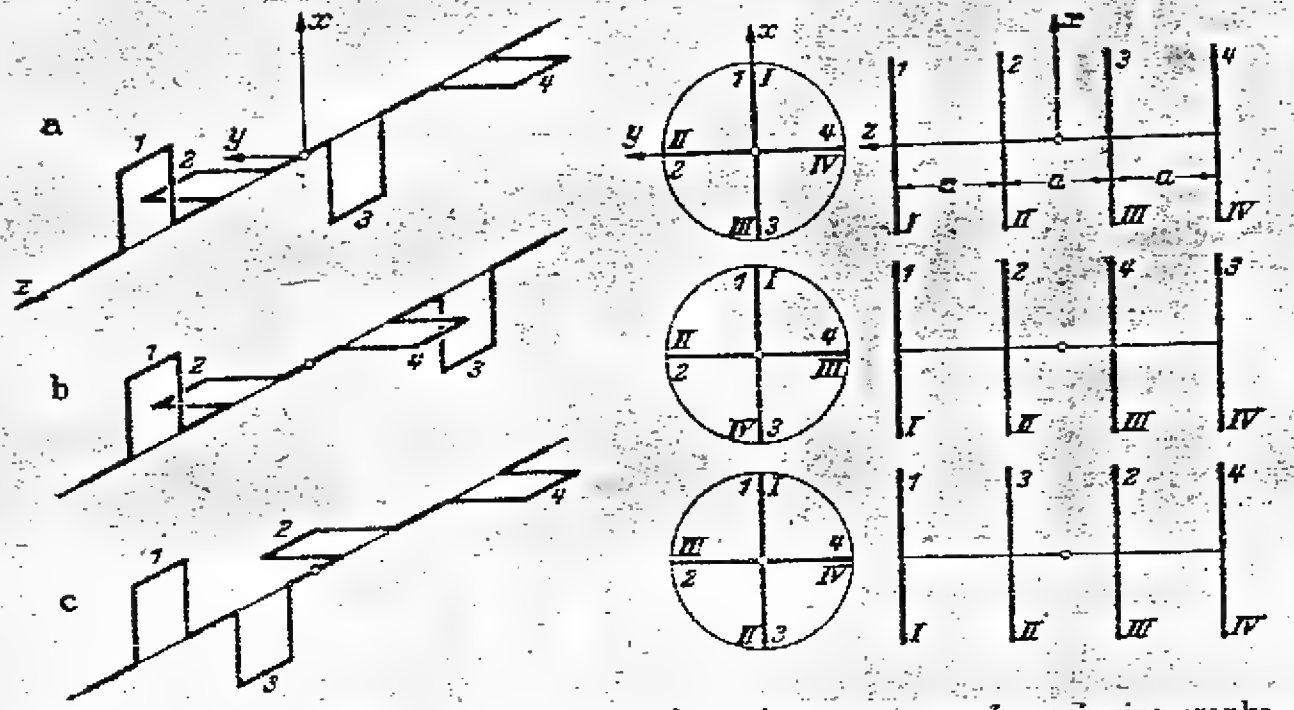


Fig. 9.—Crankshafts of four-cylinder two-cycle engines; systems of numbering cranks and cylinders.

We specify that the numbers 1 2 3 4 correspond to the positive sense of ψ, so that the firing order is the inverse, i.e. 1 4 3 2 1. Thus the angle from each crank to the following one is positive, being 90° here.

The origin of coordinates is taken again on the crank axis midway between the inner cranks. The distance between adjacent cylinder axes is a, a constant. If ψ is the rotation angle of crank 1 then the moments for the three different crank arrangements are

$$\left.\begin{aligned} \text{(a)}\quad M_x &= -\sqrt{8}\,\omega^2 aQ' \sin(\psi + 45°) - \{(\omega)\},\\ M_y &= \sqrt{8}\,\omega^2 a(Q + Q')\cos(\psi - 45°) + 2\omega^2 aQ(A_2\cos 2\psi + \\ &\qquad + A_6\cos 6\psi + \cdots) + \{(\omega)\}; \end{aligned}\right\} \tag{3}$$

$$\left.\begin{aligned} \text{(b)}\quad M_x &= -\sqrt{10}\,\omega^2 aQ' \sin(\psi + \beta) - \{(\omega)\},\\ M_y &= \sqrt{10}\,\omega^2 a(Q + Q')\cos(\psi + \beta) + \{(\omega)\}; \end{aligned}\right\} \quad (\beta = \arctan\tfrac{1}{3} = 18{\cdot}4°) \tag{4}$$

$$\left.\begin{aligned} \text{(c)}\quad M_x &= -\sqrt{2}\,\omega^2 aQ' \sin(\psi + 45°) - \{(\omega)\},\\ M_y &= \sqrt{2}\,\omega^2 a(Q + Q')\cos(\psi - 45°) + 4\omega^2 aQ(A_2\cos 2\psi + \\ &\qquad + A_6\cos 6\psi + \cdots) + \{(\omega)\}. \end{aligned}\right\} \tag{5}$$

Since the first-order moments are generally the most serious the arrangement (*c*) is normally preferred to (*a*) or (*b*). However, it should be noticed that arrangement (*b*), which is the worst in respect to first-order moments, excites no moments of higher order and is, therefore, to be preferred when higher-order resonance is feared. Moreover, since in practice it is impossible to make $Q = 0$, the arrangement (*b*) affords the only possibility of eliminating the frequently undesirable pitching moment M_y. This is accomplished by means of crank counterweights to make $Q + Q' = 0$. (See Sect. 6 and the corresponding discussion in Sect. 8.)

The forces and moments given by equations (1) to (5) can be obtained very simply by graphical means. We define the unit vectors $\mathbf{e}_k^i$ directed

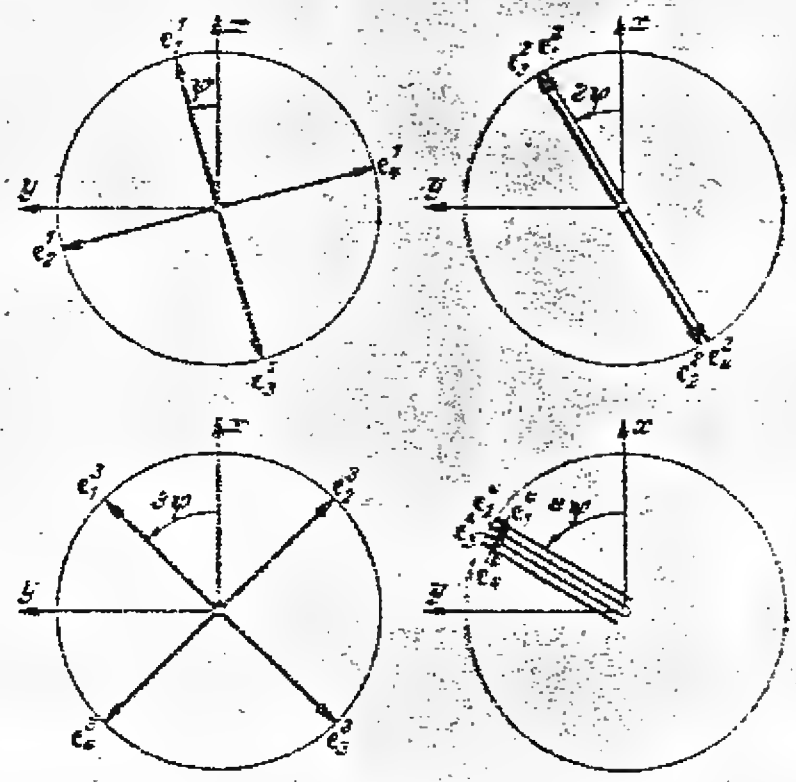


Fig. 10.—Unit vectors $\mathbf{e}_k^i$ in the crank diagram.

radially outward from the crank axis in the crank diagram. The vectors $\mathbf{e}_1^1$, $\mathbf{e}_2^1$, $\mathbf{e}_3^1$, $\mathbf{e}_4^1$ are in the directions of the cranks, having the azimuths ψ, $\psi + 90°$, $\psi + 180°$, $\psi + 270°$. The vectors $\mathbf{e}_1^2$, $\mathbf{e}_2^2$, $\mathbf{e}_3^2$, $\mathbf{e}_4^2$ have exactly doubled azimuths, $\mathbf{e}_1^3$, $\mathbf{e}_2^3$, $\mathbf{e}_3^3$, $\mathbf{e}_4^3$ tripled azimuths, etc., as in Fig. 10. (The upper indices used here must not be confused with exponents.) For example, using (5, 1) we find the total longitudinal force for any of these crank arrangements:

$$X = \omega^2[(Q + Q')\Sigma_x \mathbf{e}_1^1 + Q(A_2\Sigma_x \mathbf{e}_1^2 - A_4\Sigma_x \mathbf{e}_1^4 + A_6\Sigma_x \mathbf{e}_1^6 - + \cdots) + \\ + \omega[(Q + Q')\Sigma_y \mathbf{e}_1^1 + Q(\tfrac{1}{2}A_2\Sigma_y \mathbf{e}_1^2 - \tfrac{1}{4}A_4\Sigma_y \mathbf{e}_1^4 + \tfrac{1}{6}A_6\Sigma_y \mathbf{e}_1^6 - + \cdots)].$$

In this equation Σ_x and Σ_y represent the summations of the x- and y-components of the vectors that are to be added. However, as can be seen from Fig. 10

$$\Sigma \mathbf{e}_1^1 = 0, \qquad \Sigma \mathbf{e}_1^2 = 0,$$
$$\Sigma \mathbf{e}_1^3 = 0, \qquad \Sigma \mathbf{e}_1^4 = 4,$$

or, generally,

$$\sum \mathfrak{e}_i^j = \begin{cases} 6 & \text{for } j = 4, 8, 12, \dots \\ 4 & \text{for } j = 4, 8, 12, \dots \end{cases} \tag{6}$$

Since analogous expressions hold for Y and for M_z formulæ (1) and (2) are easily obtained.

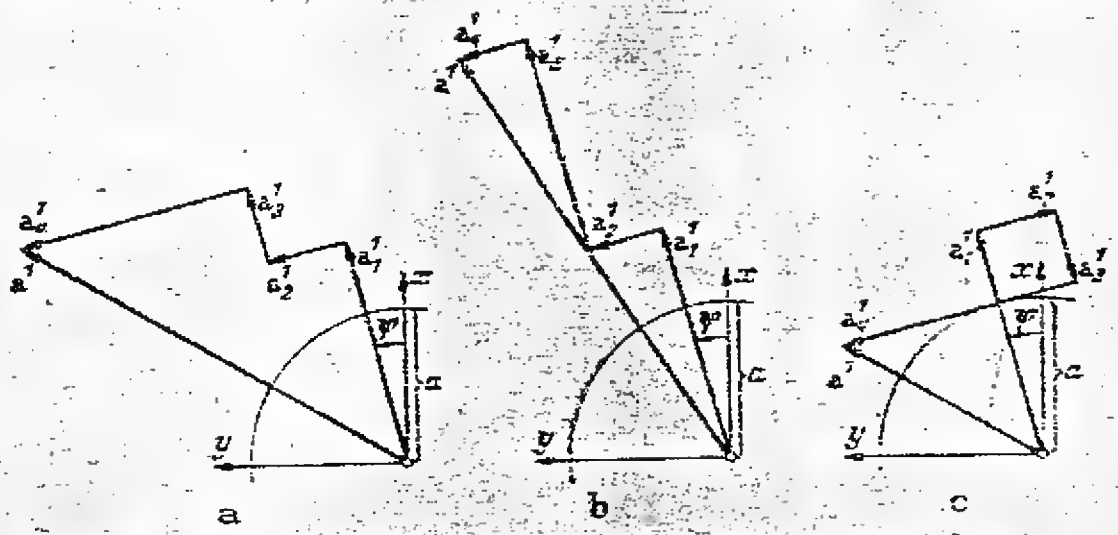


Fig. 11.—Vector diagrams of $\mathfrak{a}_2^j$ for crank arrangements a, b, c.

Letting a_1, a_2, a_3, and a_4 be the distances of the four cylinders from the x-axis, taken as positive in the positive z-direction, we define other vectors

$$\mathfrak{a}_i^j = a_i \mathfrak{e}_i^j \tag{7}$$

by means of which the complete M_z and M_y can be expressed:

$$\left.\begin{aligned} M_z &= -\Sigma a_k Y_k = -\omega^2 Q' \Sigma_r \mathfrak{a}_r^1 + \omega Q' \Sigma_r \mathfrak{a}_r^1 \\ M_y &= \Sigma a_k X_k = \omega^2 [(Q - Q') \Sigma_r \mathfrak{a}_r^1 + Q(A_2 \Sigma_r \mathfrak{a}_r^2 - A_4 \Sigma_r \mathfrak{a}_r^4 + A_6 \Sigma_r \mathfrak{a}_r^6 - + \cdots)] + \\ &\quad + \omega [(Q + Q') \Sigma_r \mathfrak{a}_r^1 - Q(\tfrac{1}{2} A_2 \Sigma_r \mathfrak{a}_r^2 - \tfrac{1}{4} A_4 \Sigma_r \mathfrak{a}_r^4 + \tfrac{1}{6} A_6 \Sigma_r \mathfrak{a}_r^6 - + \cdots)]. \end{aligned}\right\} \tag{8}$$

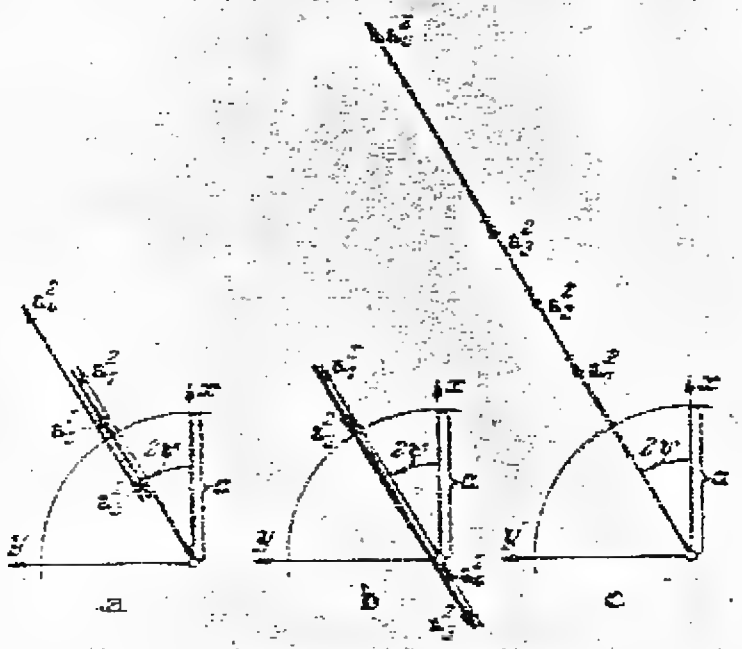


Fig. 12.—Vector diagrams of $\mathfrak{a}_2^j$ for crank arrangements a, b, c.

In Figs. 11 to 13 are given the vector sums $\mathfrak{a}^j = \Sigma \mathfrak{a}_i^j$ $(j = 1, 2, 4)$. Equations (3) to (5) follow from (8) with no difficulty.

By means of the vector diagrams shown here we can very quickly decide

whether or not the moments M_z and M_y can be reduced by changing the spacings between cylinders. Since in any case the spacings are made as small as possible for reasons of construction, and since only symmetrical arrangements are considered, this question is simply whether or not M_z and M_y become smaller when we increase the spacing of either the inner

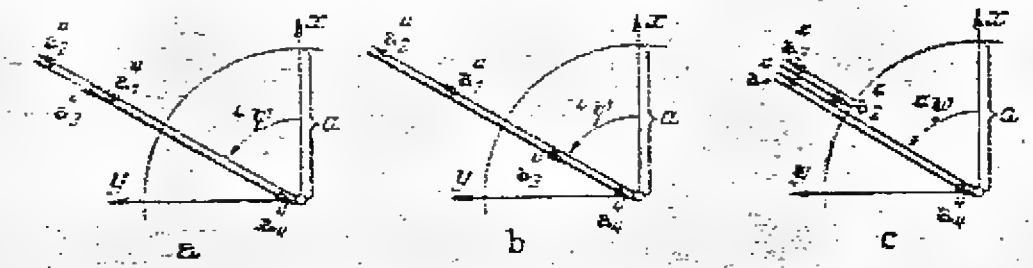


Fig. 13.—Vector diagrams of $\mathfrak{a}_r^1$ for crank arrangements a, b, c.

or the outer cylinders. From Fig. 11 it is obvious that for arrangements (*a*) and (*b*) the answer is negative, since an increase in any of the four vectors $\mathfrak{a}_k^1$ increases the sum $\mathfrak{a}^1$. In case (*c*) an increase in the vectors $\mathfrak{a}_2^1$ and $\mathfrak{a}_3^1$ (increasing the spacing of the inner cylinders) decreases the sum $\mathfrak{a}^1$, provided the distances of the outer cylinders from the origin remain unchanged. However, that is impossible if the outer cylinders are already as close as possible to the inner ones. In fact, the vectors $\mathfrak{a}_1^1$ and $\mathfrak{a}_4^1$ must then increase by the same amount as $\mathfrak{a}_2^1$ and $\mathfrak{a}_3^1$, giving no change in the resultant $\mathfrak{a}^1$ or in the pitching and rolling moments of the first order. However, it is worth noting that for moments of the first order the arrangement (*c*) becomes no

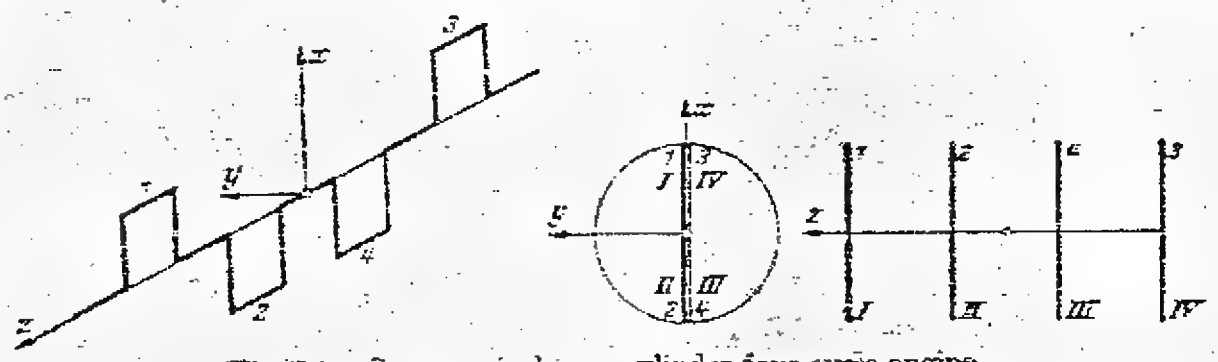


Fig. 14.—Symmetrical four-cylinder four-cycle engine.

worse when the cylinders I and II are separated from III and IV. The moments of higher order are increased, of course.

(b) *Four-stroke cycle.*—The four-cylinder four-stroke engine generally has a crank offset of $4 \times \pi/4 = 180^\circ$ and consequently, in respect to inertia forces and moments, it is equivalent to two two-cylinder two-stroke engines built together. Therefore

$$Z = 4\omega^2 Q(A_2 \cos 2\varphi - A_4 \cos 4\varphi + A_6 \cos 6\varphi - + \cdots + ((\omega)). \quad (9)$$

$$Y = 0, \quad (10)$$

$$M_z = -4\omega R'.$$

Of the three possible crank arrangements, the symmetric one shown in Fig. 14 is by far the best, since with it the pitching and yawing moments

due to mechanisms 1 and 2 are balanced by those due to 4 and 3, so that

$$M_z = M_y = 0. \tag{11}$$

The longitudinal-force components of second, fourth, etc., order and the unsteady part of the rolling moment are not eliminated. (However, see Sect. 15.)

10. Inertia Forces and Rolling Moment for the n-Cylinder Engine.—(a) *Two-stroke cycle.*—In the normal n-cylinder two-stroke engine any two consecutively firing cranks are separated by the angle $2\pi/n$. (The firing order is $\ldots 1\ n\ n-1 \ldots 3\ 2\ 1 \ldots$.) The rotation of crank 1 is ψ and that of crank k is

$$\psi_k = \psi + (k-1)\frac{2\pi}{n} \qquad (k = 1, 2, \ldots, n). \tag{1}$$

From (5, 1) to (5, 3) the total longitudinal force, the transverse force, and the rolling moment are

$$\left.\begin{aligned} X &= \Sigma X_k = \omega^2[(Q+Q')\Sigma\cos\psi_k + Q(A_2\Sigma\cos 2\psi_k - A_4\Sigma\cos 4\psi_k + \\ &\qquad + A_6\Sigma\cos 6\psi_k - + \cdots) + ((\omega)), \\ Y &= \Sigma Y_k = \omega^2 Q'\Sigma\sin\psi_k + ((\omega)), \end{aligned}\right\} \tag{2}$$

$$\left.\begin{aligned} M_z &= \Sigma M_{zk} = \omega^2 R(C_1\Sigma\sin\psi_k - C_3\Sigma\sin 3\psi_k + C_5\Sigma\sin 5\psi_k - + \cdots) + \\ &\qquad + ((\omega)) - n\omega R'. \end{aligned}\right\} \tag{3}$$

The summations indicated here are from $k = 1$ to $k = n$ and are easily evaluated with the help of the following identities (obtained from de Moivre's Theorem [1]):

$$\left.\begin{aligned} \sum_{k=1}^{n}\cos[\alpha + (k-1)\beta] &= \frac{\sin\frac{n\beta}{2}\cos\left(\alpha + \frac{n-1}{2}\beta\right)}{\sin\frac{\beta}{2}}, \\ \sum_{k=1}^{n}\sin[\alpha + (k-1)\beta] &= \frac{\sin\frac{n\beta}{2}\sin\left(\alpha + \frac{n-1}{2}\beta\right)}{\sin\frac{\beta}{2}}, \end{aligned}\right\} \quad (\beta = 2\pi, 4\pi, 6\pi, \ldots). \tag{4a}$$

together with the obvious identities

$$\left.\begin{aligned} \sum_{k=1}^{n}\cos[\alpha + (k-1)\beta] &= n\cos\alpha \\ \sum_{k=1}^{n}\sin[\alpha + (k-1)\beta] &= n\sin\alpha \end{aligned}\right\} \quad (\beta = 2\pi, 4\pi, 6\pi, \ldots). \tag{4b}$$

[1] See E. Hammer, "Lehr- und Handbuch der ebenen und sphärischen Trigonometrie", p. 216, 3rd. ed., Stuttgart, 1907.

If we let $\alpha = j\psi$ and $\beta = 2j\pi/n$ with $j = 1, 2, \ldots, n, \ldots$ and introduce (1), these expressions become

$$\left.\begin{aligned} \sum_{k=1}^{n} \cos j\psi_k &= \begin{cases} 0 & \text{for } j \neq n, 2n, 3n, \ldots \\ n \cos j\psi & \text{for } j = n, 2n, 3n, \ldots \end{cases} \\ \sum_{k=1}^{n} \sin j\psi_k &= \begin{cases} 0 & \text{for } j \neq n, 2n, 3n, \ldots \\ n \sin j\psi & \text{for } j = n, 2n, 3n, \ldots \end{cases} \end{aligned}\right\} \quad (4c)$$

Thus if n is assumed greater than 1 (eliminating the one-cylinder engine discussed in Sect. 6) the result is

$$\left.\begin{aligned} X &= n\omega^2 Q(\pm A_n \cos n\psi - A_{2n} \cos 2n\psi \pm A_{3n} \cos 3n\psi - \ldots) + f(\omega)) \text{ for } n = \begin{cases} 2, 6, 10, \ldots \\ 4, 8, 12, \ldots \end{cases} \\ X &= n\omega^2 Q(A_{2n} \cos 2n\psi - A_{4n} \cos 4n\psi + A_{6n} \cos 6n\psi - + \cdots) + ((\omega)) \text{ for } n = 3, 5, 7, \ldots \end{aligned}\right\} \quad (5)$$

$$Y = 0 \text{ for } n = 2, 3, 4, 5, \ldots \quad (6)$$

$$\left.\begin{aligned} M_x &= -n\omega R' \text{ for } n = 2, 4, 6, \ldots \\ M_z &= \mp n\omega^2 R(C_n \sin n\psi - C_{2n} \sin 3n\psi + C_{5n} \sin 5n\psi - + \cdots) \mp \\ &\quad \mp ((\omega)) - n\omega R' \text{ for } n = \begin{cases} 3, 7, 11, \ldots \\ 5, 9, 13, \ldots \end{cases} \end{aligned}\right\} \quad (7)$$

The following conclusions are drawn:

(1) The maximum longitudinal force (i.e. the first that does not vanish) is of order n in engines with an even number of cylinders and of order $2n$ with an odd number of cylinders. Since the coefficients A_n diminish rapidly with an increase in the order number n, it follows that the engine with an odd number of cylinders is superior to that with an even number of cylinders from the point of view of balancing longitudinal forces.

(2) The transverse force in the multi-cylinder engine is completely balanced.

(3) Apart from the unsteady component of zero order (which is always present) the rolling moment for an even number of cylinders is completely balanced, and for an odd number of cylinders it is of order n at least, but is easily eliminated by making $R = 0$ or $k'^2 = s'(l - s')$.

(4) If forces of sixth and higher order can be ignored, all engines with five or more cylinders can be considered balanced.

The table on p. 24, which includes for completeness the one-cylinder engine previously excluded, gives the orders of the unbalanced forces and rolling moments. The moments can be eliminated by making $R = 0$.

The unit vectors $\mathfrak{e}_k^j$ with azimuth $j\psi_k$ defined in Sect. 9 can be used here to develop these results graphically (as was done for the case $n = 4$). The summations appearing in (2) and (3) are the x- and y-components of the vector sums $\Sigma \mathfrak{e}_k^j$. It is obvious immediately that any vector set $\mathfrak{e}_1^j, \mathfrak{e}_2^j, \mathfrak{e}_3^j \ldots \mathfrak{e}_n^j$ always forms a symmetrical radial pattern (n-fold, $\frac{1}{2}n$-fold, $\frac{1}{3}n$-fold, etc., according to whether or not $n/2$, $n/3$, etc., is a whole number) and therefore has a sum of zero, except when $j = n, 2n, 3n \ldots$, in which case all vectors

have the same direction and combine to give a vector of length n. Fig. 15 shows the vectors e for five- and six-cylinder engines.

Unbalanced Forces ● and Rolling Moments ○ in Two-cycle Engines

No. cyl. \ Order	1	2	3	4	5	6	7	8	9	10	11	12
1	●○	●	○	●	○	●	○	●	○	●	○	●
2		●		●		●		●		●		●
3			○			●			○			●
4				●				●				●
5					○					●		
6						●						●
7							○					
8								●				
9									○			
10										●		
11											○	
12												●

(b) *Four-stroke cycle.*—In the n-cylinder four-stroke engine we must distinguish between even and odd numbers of cylinders. When n is even

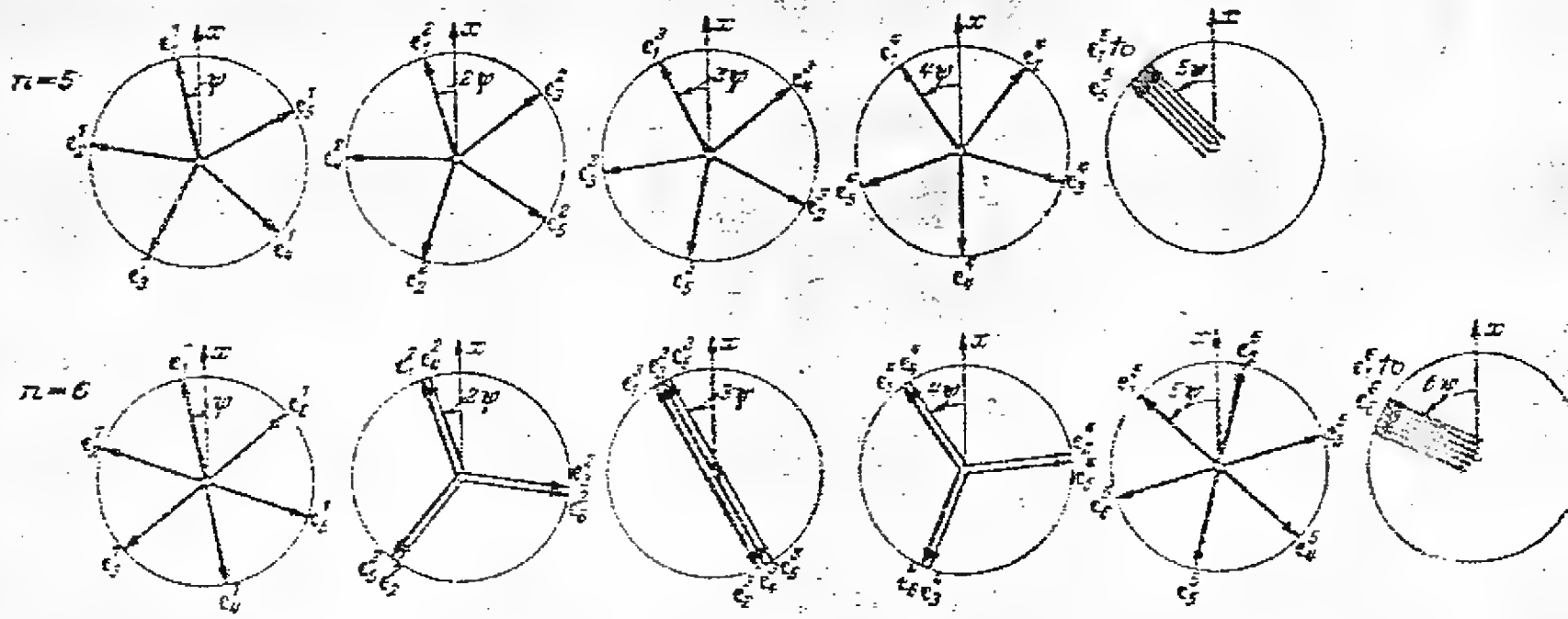


Fig. 15.—Unit vectors e_p^r in the crank diagrams of five- and six-cylinder engines.

the usual crank offset angle is $4\pi/n = 2\pi/\frac{1}{2}n$ and is the same as that of a two-stroke engine with $\frac{1}{2}n$ cylinders. Thus the four-stroke engine with an even number of cylinders is equivalent to two two-stroke engines, each

with half as many cylinders, built together. Consequently

$$X = n\omega^2 Q\left(\mp A_{\frac{n}{2}} \cos\frac{n}{2}\varphi - A_n \cos n\varphi \pm A_{\frac{3n}{2}} \cos\frac{3n}{2}\varphi - \cdots\right) + ((\omega)) \text{ for } n = \begin{cases} 4, 12, 20, \ldots \\ 8, 16, 24, \ldots \end{cases} \quad (8)$$

$$X = n\omega^2 Q(A_n \cos n\varphi - A_{2n} \cos 2n\varphi + A_{3n} \cos 3n\varphi - + \cdots) + ((\omega)) \text{ for } n = 6, 10, 14, \ldots$$

$$Y = 0 \text{ for } n = 4, 6, 8, \ldots \quad (9)$$

$$M_z = -n\omega R' \text{ for } n = 4, 8, 12, \ldots$$

$$M_x = \mp n\omega^2 R\left(C_{\frac{n}{2}} \sin\frac{n}{2}\varphi - C_{\frac{3n}{2}} \sin\frac{3n}{2}\varphi + C_{\frac{5n}{2}} \sin\frac{5n}{2}\varphi - + \cdots\right) \mp \mp((\omega)) - n\omega R' \text{ for } n = \begin{cases} 6, 14, 22, \ldots \\ 10, 18, 26, \ldots \end{cases} \quad (10)$$

On the other hand, if n is odd, only the method of numbering and not the appearance of the crank diagram distinguishes the four- from the two-stroke engine. (If the 1st, 3rd, 5th . . . nth rays in the regular n-fold radial vector diagram for the two-stroke engine are taken as the 1st, 2nd, 3rd, . . . $\frac{1}{2}(n+1)$th rays for the four-stroke engine, the omitted 2nd, 4th, . . . $(n-1)$th rays correspond exactly to the $\frac{1}{2}(n+3)$, $\frac{1}{2}(n+5)$, . . . nth rays for the four-stroke engine. The firing order is . . . 1 $n-1$ $n-3$ . . . 4 2 n $n-2$ . . . 3 1 . . .). Therefore, from the results already found for the two-stroke engine, we get

$$X = n\omega^2 Q(A_{2n} \cos 2n\varphi - A_{4n} \cos 4n\varphi + A_{6n} \cos 6n\varphi - + \cdots) + + ((\omega)) \text{ for } n = 3, 5, 7, \ldots \quad (8a)$$

$$Y = 0 \text{ for } n = 3, 5, 7, 9, \ldots \quad (9a)$$

$$M_z = \mp n\omega^2 R(C_n \sin n\varphi - C_{3n} \sin 3n\varphi + C_{5n} \sin 5n\varphi - + \cdots) \mp \mp((\omega)) - n\omega R' \text{ for } n = \begin{cases} 3, 7, 11, \ldots \\ 5, 9, 13, \ldots \end{cases} \quad (10a)$$

The conclusions are:

(1) In engines whose cylinder number is divisible by four the maximum longitudinal force is of order $\frac{1}{2}n$, with an even number of cylinders not divisible by four it is of order n, and with an odd number of cylinders of order $2n$. For balancing of longitudinal forces, engines with an odd number of cylinders are better than those with an even number, and of the latter those with numbers not divisible by four are superior.

(2) The transverse force in multi-cylinder engines is completely balanced.

(3) Except for the unsteady moment of zero order which is always present, the rolling moment is completely balanced in engines whose cylinder number is divisible by four, is of order $\frac{1}{2}n$ or higher with an even number not divisible by four, and is of order n or higher with an odd number of cylinders. The rolling moment can be completely eliminated by making $R = 0$ or $k'^2 = s'(l - s')$.

The table on p. 26 shows the orders of the unbalanced forces and rolling moments. The results can also be represented graphically by means of the unit vector diagram.

Unbalanced Forces ● and Rolling Moments ○ in Four-cycle Engines

No. cyl. \ Order	1	2	3	4	5	6	7	8	9	10	11	12
1	●○	●	○	●	○	●	○	●	○	●	○	●
2	●○	●	○	●	○	●	○	●	○	●	○	●
3			○			●			○			●
4		●		●		●		●		●		●
5					○					●		
6			○			●			○			●
7							○					
8				●				●				●
9									○			
10					○					●		
11											○	
12						●						●

11. Yawing and Pitching Moments in the n-Cylinder Engine.—The general determination of these moments, even in engines with normal crank offsets, is a complicated problem.[1] Since there is no loss of generality in assigning crank 1 always to cylinder I, as was mentioned in Sect. 9, in arranging the remaining 2, 3, . . . n cranks among the II, III, . . . N cylinders the number of possibilities equals the number of permutations of the numbers 2, 3, . . . n, i.e. $(n-1)!$. Of course, if $n > 2$ half of these arrangements are mirror images of the others, so that the number of dynamically different engines amounts to only $\frac{1}{2}(n-1)!$. In the four-cylinder engine this is 3, in the five-cylinder it is 12, in the six-cylinder 60, and in the 12-cylinder nearly 20,000,000. Consequently, we must make a selection.

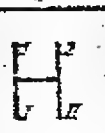
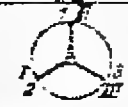
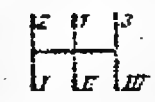
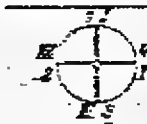
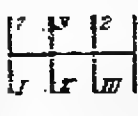
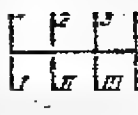
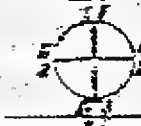
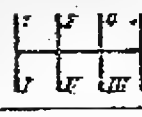
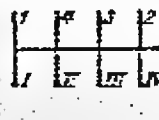
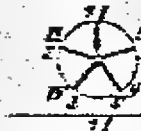
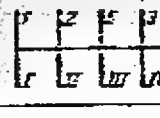
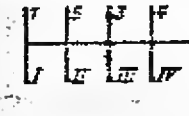
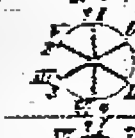
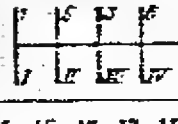
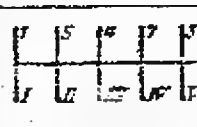
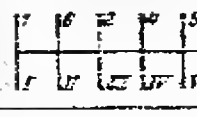
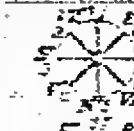
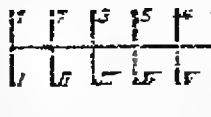
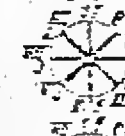
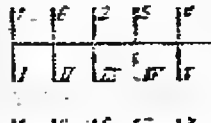
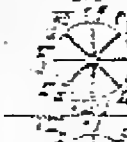
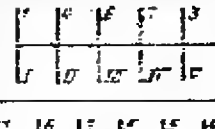
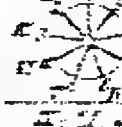
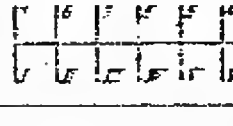
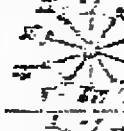
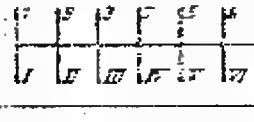
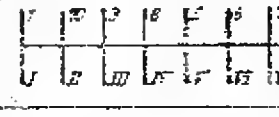
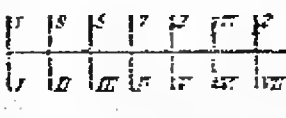
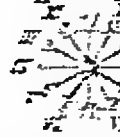
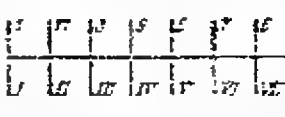
(a) *Two-stroke engine.*—If the origin of coordinates is taken on the crank axis midway between the outer cylinders, and if a_k is the distance of the kth cylinder from the origin (positive in the positive z-direction), the yawing and pitching moments are found from (10, 2) to be

$$\left.\begin{aligned} M_x &= -\Sigma a_k Y_k = -\omega^2 Q' \Sigma a_k \sin\psi_k - ((\omega)). \\ M_z &= \Sigma a_k X_k = \omega^2[(Q+Q')\Sigma a_k \cos\psi_k + Q[A_2 \Sigma a_k \cos 2\psi_k - A_4 \Sigma a_k \cos 4\psi_k + \\ &\qquad + A_6 \Sigma a_k \cos 6\psi_k - + \cdots)] + ((\omega)). \end{aligned}\right\} \quad (1)$$

These expressions can always be put into the following form in which, as before, a is the distance between any two adjacent cylinders and ψ is the

[1] H. Schrön, "Kurbelwellen mit kleinsten Massenmomenten", Berlin, 1932.

Coefficients for Yawing and Pitching Moments in Two-cycle Engines

Crank Diagram	Crank and Cylinder Arrangements	a_1 (ε_1)	a_2 (β_2)	$a_{2,1}$ ($\varepsilon_{2,1}$)	a_2 (β_2)
		1·000 (0°)	0	0	0
		1·732 (+90°)	1·732 (−90°)	1·732 (+90°)	0
		1·414 (−45°)	4·000 (0°)	0	4·000 (0°)
		2·828 (+45°)	2·000 (0°)	0	2·000 (0°)
		3·162 (+18·4°)	0	0	0
		0·449 (+54·0°)	4·980 (+18·0°)	0·449 (+126·0°)	0·449 (+54·0°)
		4·980 (+18·0°)	0·449 (−54·0°)	4·980 (+162·0°)	4·980 (+18·0°)
		0	3·464 (−30·0°)	3·464 (+150·0°)	0
		2·464 (−30·0°)	0	0	0
		0·076 (−141·4°)	9·149 (−15·9°)	3·781 (−115·7°)	0·076 (+141·4°)
		0·267 (−64·3°)	1·006 (+39·0°)	9·845 (+167·2°)	0·267 (−64·3°)
		0·448 (−67·5°)	0	16·000 (180°)	0
		0·448 (−67·5°)	11·314 (+45·0°)	0	11·314 (−45·0°)
		1·405 (+17·1°)	0	0	0
		0·194 (−70·0°)	0·548 (+50·0°)	16·330 (−130·0°)	1·732 (−30·0°)
		0	0·898 (−54·0°)	9·960 (−162·0°)	9·960 (−18·0°)
		0·153 (−73·6°)	0·382 (+57·3°)	2·636 (−133·5°)	24·436 (−8·2°)
		0	6·000 (−60·0°)	3·464 (−150°)	36·000 (0°)
		0·277 (−75·0°)	0	0	36·000 (0°)

rotation of crank 1:

$$M_x = -\alpha_1 \omega^2 a Q' \sin(\psi + \beta_1) - ((\omega)), \tag{2}$$

$$M_y = \alpha_1 \omega^2 a (Q + Q') \cos(\psi + \beta_1) + \omega^2 a Q [\alpha_2 A_2 \cos(2\psi + \beta_2) + \alpha_4 A_4 \cos(4\psi + \beta_4) + \alpha_6 A_6 \cos(6\psi + \beta_6) + \cdots] + ((\omega)). \tag{3}$$

Here α_1, α_2, α_4, α_6, ... are dimensionless quantities which, with Q and Q' and the coefficients A_2, A_4, A_6, ... (Sect. 4), determine the magnitudes of the moments of the various orders. On the other hand, the phase angles β_1, β_2, β_4, β_6, ... are usually unimportant (however, see Sect. 15). The quantities α_j, β_j can be calculated without difficulty for any configuration. For example, by making use of (10, 1) we find for the terms of first order, as can be shown by comparing (1) with (2) and (3), that

$$\alpha_1 a \cos\beta_1 = \sum a_k \cos\frac{2\pi(k-1)}{n}, \qquad \alpha_1 a \sin\beta_1 = \sum a_k \sin\frac{2\pi(k-1)}{n}.$$

Hence

$$\left.\begin{aligned} \alpha_1^2 &= \left[\sum \frac{a_k}{a} \cos\frac{2\pi(k-1)}{n}\right]^2 + \left[\sum \frac{a_k}{a} \sin\frac{2\pi(k-1)}{n}\right]^2, \\ \tan\beta_1 &= \left[\sum \frac{a_k}{a} \sin\frac{2\pi(k-1)}{n}\right] \Big/ \left[\sum \frac{a_k}{a} \cos\frac{2\pi(k-1)}{n}\right]. \end{aligned}\right\} \tag{4}$$

Similar expressions hold for the remaining α_j, β_j. The table on p. 27 gives these coefficients for the best arrangements of two- to twelve-cylinder engines. "Good" arrangements are considered to be those in which at least one of the three coefficients α_1, α_2, α_4 remains very small or is zero, since then the moment of the corresponding order is also small or zero. (In this table, for $n = 3$ the centre crank is numbered 1, to agree with Sect. 8.)

Of special importance are arrangements for which $\alpha_1 = 0$, since not only M_x but also the first-order M_y then vanish. This case can be realized for engines with 6, 10, or 12 cylinders. On the other hand, if $\alpha_1 \neq 0$ we may decide whether to make $Q' = 0$, eliminating the yawing moment M_z; to make $Q + Q' = 0$, eliminating the pitching moment of the first order; or finally, to set $\varkappa Q + Q' = 0$ in order to reduce both moments, as was done in Sect. 6 for the analogous longitudinal and transverse forces in the one-cylinder engine. When there are no intermediate bearings, and adjacent crank webs form a single piece normal to the crank axis, we may replace the two counterweights indicated for two adjacent interior webs by a single weight that is statically equivalent to the two in respect to centrifugal force.

(b) *Four-stroke engine.*—In the n-cylinder four-stroke engine we must make a distinction between engines with odd and with even numbers of cylinders. If n is even it is always possible to balance out the moments

M_x and M_y completely; the four-stroke engine is equivalent to two two-stroke engines, each with $\frac{1}{2}n$ cylinders, which are mirror images of each other (in the (x, y)-plane) and which operate in the four-stroke cycle with a phase difference of $4\pi/n$. Thus the moments M_x and M_y in the two systems are always equal and opposite, and therefore cancel each other. For $n = 4$ there is only one arrangement of this kind (Sect. 9); for $n = 6$ there is also only one; for $n = 8$ there are three; for $n = 10$ there are 12; for $n = 12$ there are 60. In general, there are $\frac{1}{2}(\frac{1}{2}n - 1)!$ such arrangements, provided $n > 4$. When there are several possible arrangements, the best is that for which each half of the engine, or more generally, each 2^kth part (where $n/2^k$ is a whole number) is as nearly balanced as possible. For example, the 12-cylinder engine is best constructed by combining two six-cylinder engines with crank arrangements 1 5 3 4 2 6 or 1 $\bar{5}$ 3 6 2 4.

On the other hand, if n is odd, exactly the same relations exist as in the two-stroke engine since both crank diagrams are identical. Therefore, the previous table can be used for all odd values of n in four-stroke engines.

§ 3. Engines with Special Features

12. Unequal Crank Offsets.—Sometimes there are good reasons for not using the standard crank offset angles ($2\pi/n$ for two-stroke, $4\pi/n$ for four-stroke engines). We shall consider three examples that may be encountered.

(a) *Two-cylinder engine with 90° crank angle.*—If the cranks in a two-cylinder engine are offset by 90° (instead of 180° for two-stroke or 0° for four-stroke engines) the vector diagram (Fig. 16) shows that the longitudinal

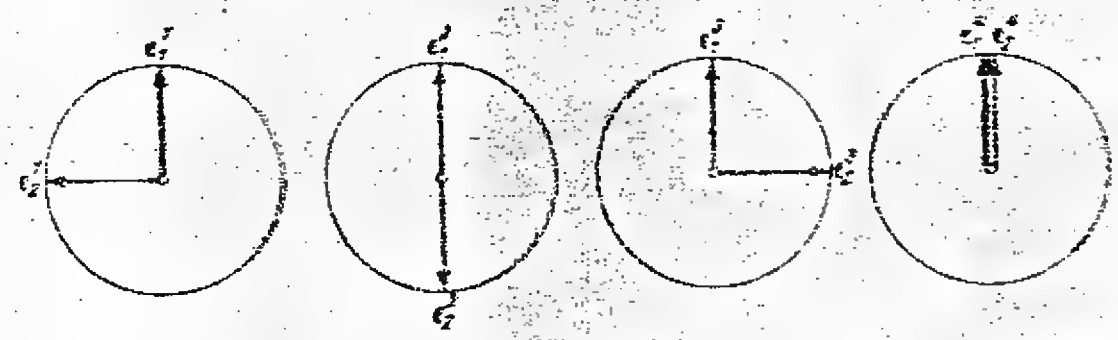


Fig. 16.—Unit vectors $\mathfrak{e}_x^j$ in the crank diagram of the two-cylinder engine with 90° crank offset.

forces of order, two, six, etc., are missing, in contrast to the normal two-cylinder engine (Sect. 7). On the other hand, we must accept the fact that forces of the first order will occur here that are not present in the normal engine. In respect to inertia force balancing, the 90° crank angle is advantageous only in the four-stroke engine; in two-stroke double-acting engines it has the advantage of constant firing intervals.

(b) *Four-cylinder engine with 135° and 45° crank angles.*—If the crankshaft of a four-cylinder engine has two pairs of opposing cranks with angles

of 135° and 45° between them (instead of 90° in the two-stroke or 180° in the four-stroke engine) the vector diagram (Fig. 17) shows that the longitudinal forces of first, fourth, twelfth, etc. order are missing, in contrast to the normal four-cylinder engine in which the fourth, twelfth, etc., are present in both two- and four-stroke versions (Sect. 9). However, we must accept the fact that the forces of second, sixth, etc., order are now present, unlike the case of the normal two-stroke engine. Only in the four-stroke cycle does this crank arrangement offer advantages in respect of inertia balancing.

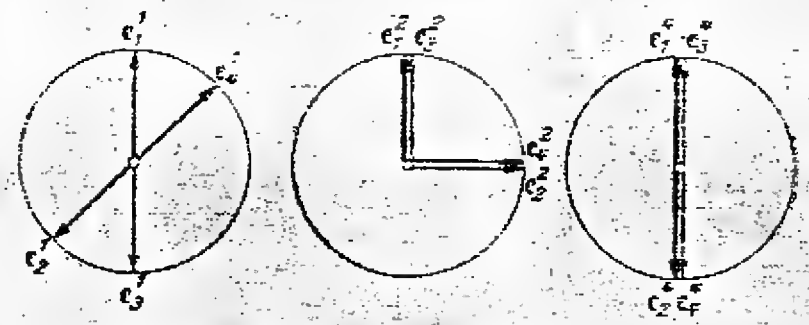


Fig. 17.—Unit vectors $\mathbf{e}_i^j$ in the crank diagram of the four-cylinder engine with 135° and 45° crank offsets.

If we compare the $\mathbf{a}_2^1$ diagrams (Fig. 18) for these crank offsets and the three arrangements of Sect. 9 with the $\mathbf{a}_2^1$ diagrams for normal engines (Fig. 11) we see that in two-stroke cycles the yawing and pitching moments of first order are smaller with the new crank angles, i.e.

(a) $\alpha_1 = 1{\cdot}531$ instead of $2{\cdot}828$,

(b) $\alpha_1 = 2{\cdot}399$ instead of $3{\cdot}162$,

(c) $\alpha_1 = 0{\cdot}765$ instead of $1{\cdot}414$.

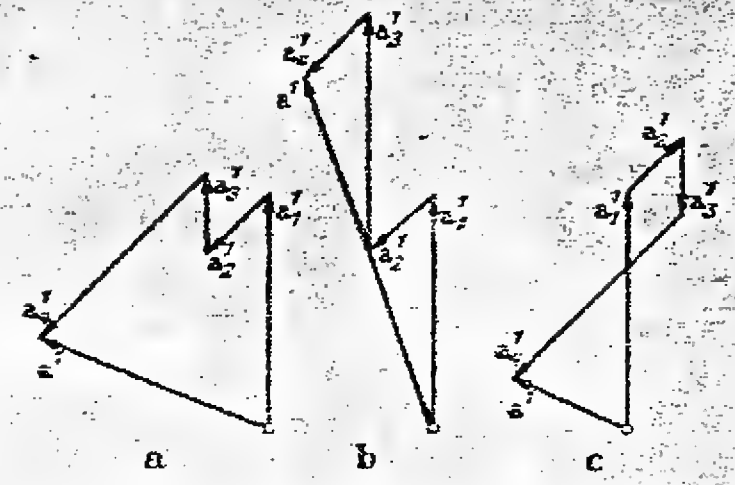


Fig. 18.—Vector diagrams of $\mathbf{a}_2^1$ for four-cylinder engines with 135° and 45° crank offsets.

Also, for moments of the second (but not the fourth) order the present engine is superior to the normal two-stroke engine, as can be shown in the same way. On the other hand, in the four-stroke cycle the normal engine (completely balanced in M_z and M_x) is better.

(c) *Six-cylinder engine with* 90° *and* 30° *crank offsets.*—With six cylinders it can be advantageous to make the crank angles alternately 90° and 30°

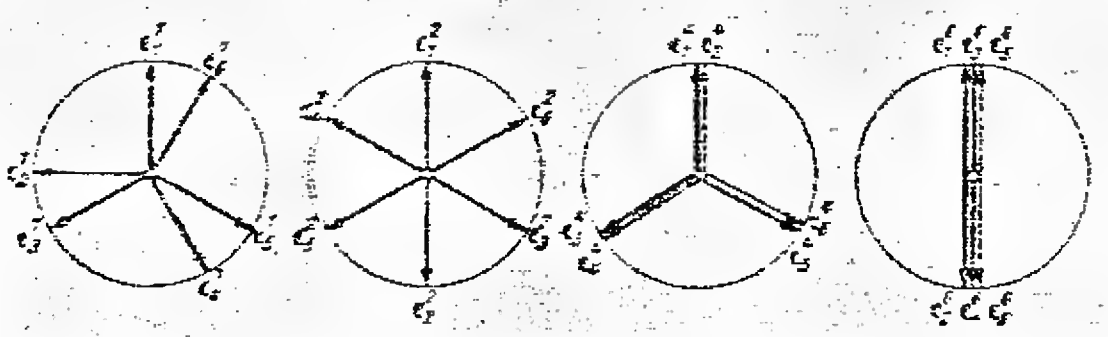


Fig. 19.—Unit vectors $\mathbf{e}_i^j$ for the six-cylinder engine with 90° and 30° crank offsets.

(instead of 60° in the two-stroke cycle or 120° in the four-stroke case). From the corresponding vector diagram (Fig. 19) it appears that the forces of

first, second, fourth, and sixth order vanish. The first non-vanishing force, as can be easily shown, is of twelfth order. Therefore, this engine is always superior to the normal two- or four-stroke engine when resonance with the force of sixth order is to be avoided. With double-acting two-stroke engines this crank arrangement offers the further advantage that all twelve firings occur at equal intervals.

On the other hand, there appears to be no way of making the first-order yawing moment or the second- or fourth-order pitching moments equal to zero in this engine, leaving it inferior in this respect to the normal engine.

13. Unequal Cylinder Spacing.—In many engines it is possible to improve the balancing of the yawing and the pitching moments by using unequal cylinder spacings. Obviously, we need consider only the case of symmetrically placed cylinders.

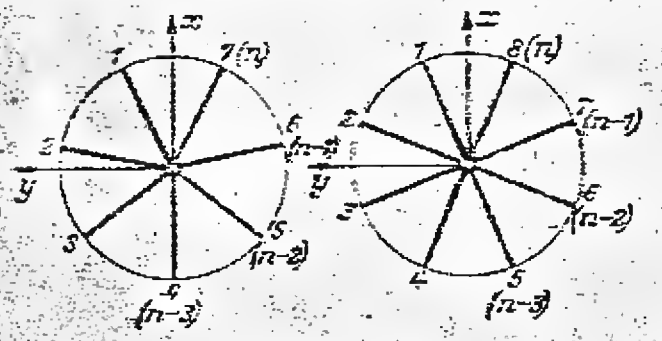


Fig. 20.—Crank diagrams for $n = 7$ and $n = 8$.

(a) *Two-stroke engine.*—According to (11, 1) the first-order moments vanish when the x- and y-components of the vector sum $\mathfrak{a}^1 = \Sigma \mathfrak{a}_k^1$ vanish:

$$\Sigma a_k \cos \psi_k = 0, \qquad \Sigma a_k \sin \psi_k = 0. \tag{1}$$

It is sufficient if this is satisfied for any one crank position since the vector $\mathfrak{a}^1$ rotates with the crank. We let $\psi = \frac{1}{2}(2\pi/n)$. (Fig. 20 shows the cases $n = 7$ and $n = 8$.) Then

$$\psi_1 = -\psi_n, \quad \psi_2 = -\psi_{n-1}, \quad \psi_3 = -\psi_{n-2}, \cdots \psi_k = -\psi_{n-k+1}, \cdots \tag{2}$$

The first of equations (1) is most simply satisfied by making the cylinder spacings as follows:

$$\left.\begin{aligned} a_1 &= -a_n, \quad a_2 = -a_{n-1}, \cdots \\ a_3 &= -a_{n-2}, \cdots a_k = -a_{n-k+1}, \cdots \end{aligned}\right\} \tag{3}$$

Thus, each pair of cylinders whose cranks are symmetric with respect to the x-axis in the crank diagram must also be symmetrically placed in the cylinder-spacing diagram (Figs. 21, 24, 25, 26, 28). Then, because $\cos \psi_k = \cos \psi_{n-k+1}$ each pair of symmetric terms is cancelled in the cosine-sum (1). Since $\sin \psi_k = -\sin \psi_{n-k+1}$ each term in the sine-sum is doubled and the second equation (1) becomes

$$a_1 \sin \psi_1 + a_2 \sin \psi_2 + \cdots + a_\varepsilon \sin \psi_\varepsilon = 0 \text{ with } \psi_k = \left(k - \frac{1}{2}\right)\frac{2\pi}{n}, \text{ where } \varepsilon = \left\{\begin{array}{l} \frac{n}{2}, \text{ if } n \text{ is even} \\ \frac{n-1}{2}, \text{ if } n \text{ is odd.} \end{array}\right\} \tag{4}$$

If the moment of order j ($= 2, 4, 6, \ldots$) is to vanish, then, from assumption (3), we must satisfy the condition

$$a_1 \sin j\psi_1 + a_2 \sin j\psi_2 + \cdots + a_\varepsilon \sin j\psi_\varepsilon = 0. \tag{5}$$

Again, the condition $\Sigma a_k \cos jv_k = 0$ is automatically satisfied, as can be easily seen by measuring crank rotations in the range from $-180°$ to $+180°$ instead of from $0°$ to $360°$, since multiplying the angles of such a symmetrical arrangement must result in another symmetrical arrangement.

If equation (4) is applied to the three- or four-cylinder engine the result is $a_1 = a_2 = a_3 = 0$ or $a_1 = -a_2 = a_3 = -a_4$, corresponding respectively to cylinders coinciding in position or placed in symmetric pairs. This agrees with the conclusions reached in Sect. 8 and 9, that in these engines the moments M_z and M_y of first order cannot be eliminated.

However, for the five-cylinder engine[1] equation (4) gives

$$a_1 : a_2 = \sin 108° : -\sin 36° = 1{\cdot}618 : -1 = 2{\cdot}618 : -1{\cdot}618.$$

In terms of the smallest cylinder spacing a we find

$$a_1 = -a_3 = 2{\cdot}618a, \quad a_2 = -a_4 = -1{\cdot}618a, \quad a_5 = 0.$$

Fig. 21 shows the cylinder arrangement in this engine for first-order balancing of yawing and pitching moments. Fig. 22 shows the arrangement for

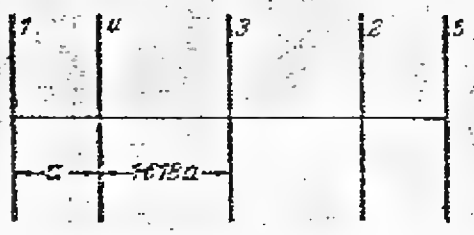


Fig. 21.—Cylinder spacing in the five-cylinder engine to give balancing of the first-order yawing and pitching moments.

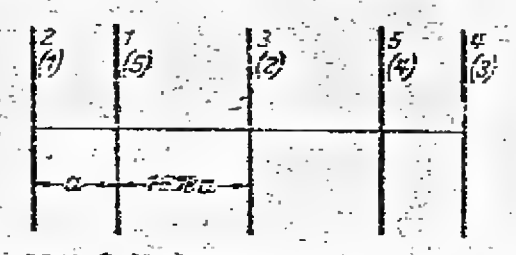


Fig. 22.—Cylinder spacing in the five-cylinder engine to give balancing of the second-order yawing and pitching moments.

second-order balancing ($j = 2$ in equation (5)). The figures in parentheses give an alternative arrangement. It is easily shown that the engine of Fig. 21 is also in balance for moments of order 4, 6, 10, 14, 16, 20, etc., while that of Fig. 22 is also in balance for moments of order 8, 10, 12, 18, 20, etc.

Since in Sect. 11 we have already found crank-cylinder arrangements for eliminating moments of first- or second-order in the six-cylinder engine with constant cylinder spacing, we can proceed immediately to consideration of the seven-cylinder engine. Condition (4) for balance of first-order moments is

$$\frac{a_3}{a_1} + 1{\cdot}248\,\frac{a_2}{a_1} + 0{\cdot}556 = 0.$$

Thus there are an infinite number of combinations that satisfy this condition; they are represented by a straight line in Fig. 23 whose coördinates are a_2/a_1 and a_3/a_1. Similarly, for second-order balancing, equation (5) with $j = 2$ gives

$$\frac{a_3}{a_1} - 0{\cdot}444\,\frac{a_2}{a_1} - 0{\cdot}802 = 0.$$

[1] H. Schnürx, "Grundgestalt der Fünfzylinder-Reihenverbrennungsmaschine für gleichförmigen und ruhigen Gang", *Motorwagen*, 31 (1928), p. 427.

This is also a straight line in Fig. 23. The intersection of the two lines is at the point

$$a_1 : a_2 : a_3 = 1 : -0{\cdot}802 : 0{\cdot}446 = 5{\cdot}058 : -4{\cdot}058 : 2{\cdot}255.$$

In terms of the smallest cylinder spacing a, we find for a seven-cylinder engine with first- and second-order moment balance

$$a_1 = -a_7 = 5{\cdot}058a, \qquad a_2 = -a_6 = -4{\cdot}058a,$$
$$a_3 = -a_5 = 2{\cdot}255a, \qquad a_4 = 0.$$

The corresponding cylinder-spacing diagram (Fig. 24) shows that the total length of such an engine would be considerable.

If we are satisfied with either first- or second-order moment balance a much smaller length of engine is possible. The best ratios are $a_1 : a_2 : a_3 = 1 : \pm 2/3 : \pm 1/3$ or $a_1 : a_2 : a_3 = 1 : \pm 1/3 : \pm 2/3$ since they result in equal cylinder spacings. The eight points that correspond to these conditions are shown by small circles in Fig. 23. It appears that two of these circles are very close to the straight lines for balanced moments in that figure. The points on the straight lines that are closest to these circles are $a_1 : a_2 : a_3 = 1 : -0{\cdot}69 : 0{\cdot}31$ for first-order balance

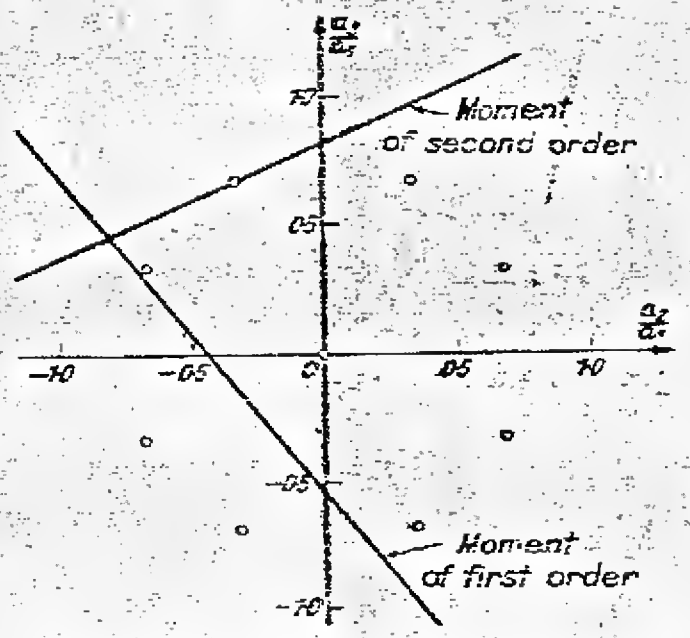


Fig. 23.—Balancing the first- and second-order moments in the seven-cylinder engine.

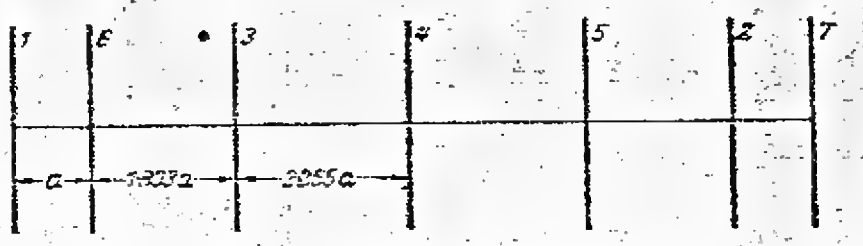


Fig. 24.—Cylinder spacing in the seven-cylinder engine to give first- and second-order moment balance.

and $a_1 : a_2 : a_3 = 1 : -0{\cdot}32 : 0{\cdot}66$ for second-order balance. The corresponding cylinder spacings are shown in Figs. 25 and 26.

Fig. 25.—Optimum cylinder spacing in the seven-cylinder engine with first-order moment balance.

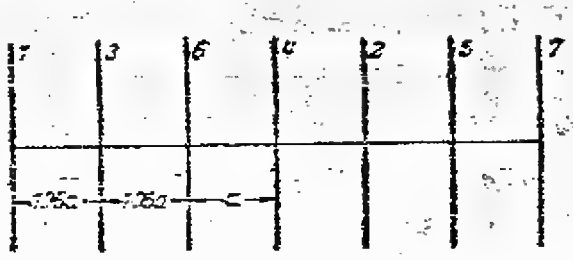


Fig. 26.—Optimum cylinder spacing in the seven-cylinder engine with second-order moment balance.

In the eight-cylinder engine the conditions (4) for first-order balancing are:

$$0{\cdot}353a_1 + 0{\cdot}924a_2 + 0{\cdot}924a_3 + 0{\cdot}383a_4 = 0$$

and those for second-order balancing are

$$a_1 + a_2 - a_3 - a_4 = 0.$$

On eliminating in one case a_4/a_1 and a_3/a_1 in the other, we get

$$\left.\begin{aligned} \frac{a_3}{a_1} + 2{\cdot}41\,\frac{a_2}{a_1} + 1{\cdot}41 = 0. \\ \frac{a_4}{a_1} - 3{\cdot}41\,\frac{a_2}{a_1} - 2{\cdot}41 = 0. \end{aligned}\right\} \qquad (6)$$

With the ratio a_2/a_1 as abscissæ and the ratios a_3/a_1 and a_4/a_1 as ordinates, equations (6) are plotted as the two straight lines of Fig. 27. In the same figure are also shown (circles) the points corresponding to various arrangements of equally spaced cylinders. Each arrangement is represented by a pair of points with the same abscissa and with opposite but unequal ordinates (a_3/a_1 and a_4/a_1). The most useful parts of the straight line are, consequently, those lying near such a pair of points. For example, if we make $a_2/a_1 = -0{\cdot}76$ we get from equation

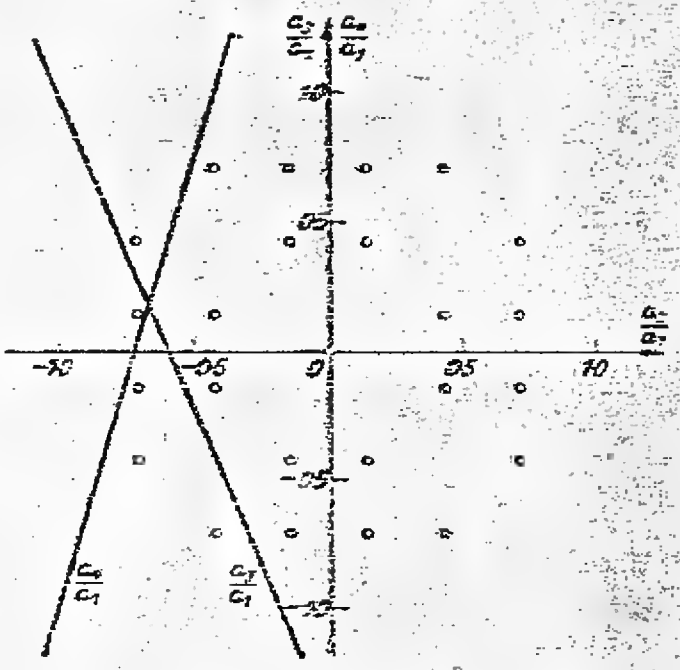


Fig. 27.—First- and second-order moment balancing in the eight-cylinder two-cycle engine.

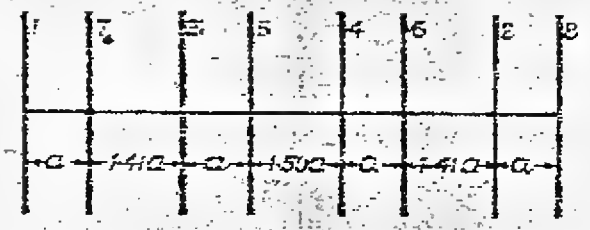


Fig. 28.—Optimum cylinder spacing in the eight-cylinder two-cycle engine with first- and second-order moment balance.

(6) $a_3/a_1 = 0{\cdot}42$ and $a_4/a_1 = -0{\cdot}18$, giving the engine represented in Fig. 28 with alternately equal cylinder spacings; the individual pairs of cylinders do not have the same spacing from each other. A simple calculation shows that fourth-order balancing is not possible with arrangements of type (3) if first- and second-order balancing is prescribed. We then arrive at the proportions $a_1 : a_2 : a_3 : a_4 = 1 : -0{\cdot}415 : -0{\cdot}415 : 1$, which would require pairs of superimposed cylinders (1, 4; 2, 3; 5, 8; 6, 7).

(b) *Four-stroke engine.*—The four-stroke engine is taken care of by observing that with an odd number of cylinders the same relations hold as with

two-stroke engines, while with an even number of cylinders, according to Sect. 11, complete moment balancing is possible with equal cylinder spacing.

14. Offset Cylinders.—The friction of the piston in the cylinder can be reduced by offsetting the cylinder a distance b in the y-direction so that the angle χ between the cylinder axis and the connecting-rod is smaller during the working stroke than during the compression stroke. For an offset mechanism [1] (Fig. 29) equations (4) are replaced by

$$x = r\cos\psi + l\cos\chi, \qquad r\sin\psi = b + l\sin\chi. \tag{1}$$

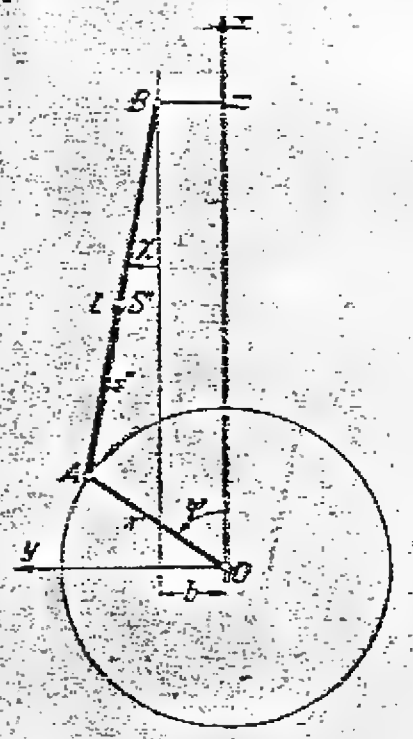


Fig. 29.—Offset reciprocating mechanism.

On introducing the crank ratio $\lambda = r/l$ and the offset ratio $\beta = b/l$ and eliminating χ, we get

$$\frac{x}{r} = \cos\psi + \frac{1}{\lambda}[1 - (\lambda\sin\psi - \beta)^2]^{1/2}. \tag{2}$$

If the right-hand side of this equation is developed in powers of λ and β and the powers of $\sin\psi$ replaced by expressions involving sine and cosine of the multiple angles, there results, as the generalization of (4, 5)

$$\left.\begin{aligned}\frac{x}{r} = {} & A_0^* + \cos\psi + A_2^*\sin\psi + \frac{1}{4}A_2^*\cos 2\psi - \frac{1}{9}A_3^*\sin 3\psi - \\ & - \frac{1}{16}A_4^*\cos 4\psi + \frac{1}{25}A_5^*\sin 5\psi + \frac{1}{36}A_6^*\cos 6\psi - - + + \cdots.\end{aligned}\right\} \tag{3}$$

The coefficients A_s^* are given accurately enough (to the fifth order of λ and β) by

$$\left.\begin{aligned}A_0^* &= A_0 - \frac{1}{2}\beta^2\left(\frac{1}{\lambda} + \frac{3}{4}\lambda + \frac{45}{64}\lambda^3\right) - \frac{1}{8}\beta^4\left(\frac{1}{\lambda} + \frac{15}{4}\lambda\right) - \frac{1}{16}\frac{\beta^6}{\lambda},\\ A_1^* &= \beta\left(1 + \frac{3}{8}\lambda^2 + \frac{15}{64}\lambda^4\right) + \frac{1}{2}\beta^3\left(1 + \frac{15}{8}\lambda^2\right) + \frac{3}{8}\beta^5,\\ A_2^* &= A_2 + \frac{3}{2}\beta^2\lambda\left(1 + \frac{5}{4}\lambda^2\right) + \frac{15}{8}\beta^4\lambda,\\ A_3^* &= \frac{9}{8}\beta\lambda^2\left(1 + \frac{15}{16}\lambda^2\right) + \frac{45}{16}\beta^3\lambda^2,\\ A_4^* &= A_4 + \frac{15}{8}\beta^2\lambda^3,\\ A_5^* &= \frac{75}{128}\beta\lambda^4,\\ A_6^* &= A_6.\end{aligned}\right\} \tag{4}$$

In these expressions the coefficients A_{2s} are given by (4, 6).

[1] R. E. Root, "Dynamics of Engine and Shaft", p. 78, Chapman and Hall, New York, 1932.

For the piston acceleration we get

$$\left.\begin{aligned} -\frac{\ddot{x}}{r} = \omega^2(&\cos\psi + A_1^*\sin\psi + A_2^*\cos 2\psi - A_3^*\sin 3\psi - A_4^*\cos 4\psi + \\ &+ A_5^*\sin 5\psi + A_6^*\cos 6\psi - - + + \cdots) + \\ + \omega\Big(&\sin\psi - A_1^*\cos\psi + \frac{1}{2}A_2^*\sin 2\psi + \frac{1}{3}A_3^*\cos 3\psi - \frac{1}{4}A_4^*\sin 4\psi - \\ &- \frac{1}{5}A_5^*\cos 5\psi + \frac{1}{6}A_6^*\sin 6\psi + - - + \cdots\Big). \end{aligned}\right\} \tag{5}$$

If we differentiate the second equation (1) with respect to time and then eliminate χ we find

$$\dot{\chi} = \omega\lambda\frac{\cos\psi}{\cos\chi} = \omega\lambda\cos\psi\,[1 - (\lambda\sin\psi - \beta)^2]^{-\frac{1}{2}},$$

which can be developed into

$$\dot{\chi} = \omega\lambda\Big(C_1^*\cos\psi - \frac{1}{2}C_2^*\sin 2\psi - \frac{1}{3}C_3^*\cos 3\psi + \frac{1}{4}C_4^*\sin 4\psi + \frac{1}{5}C_5^*\cos 5\psi - - + + \cdots\Big) \tag{6}$$

with the coefficients (up to the fifth order)

$$\left.\begin{aligned} C_1^* &= C_1 + \frac{3}{8}\beta^2\Big(1 + \frac{9}{8}\lambda^2\Big) + \frac{3}{8}\beta^4, \\ C_2^* &= \beta\lambda\Big(1 + \frac{3}{4}\lambda^2\Big) + \frac{3}{2}\beta^3\lambda, \\ C_3^* &= C_3 + \frac{27}{16}\beta^2\lambda^2, \\ C_4^* &= \frac{3}{4}\beta\lambda^3, \\ C_5^* &= C_5. \end{aligned}\right\} \tag{7}$$

Here, the coefficients C_i are the numbers calculated according to (4, 10). Therefore

$$\left.\begin{aligned} \ddot{\chi} = -\omega^2\lambda\,(&C_1^*\sin\psi + C_2^*\cos 2\psi - C_3^*\sin 3\psi - C_4^*\cos 4\psi + C_5^*\sin 5\psi + - - + \cdots) + \\ + \omega\lambda\Big(&C_1^*\cos\psi - \frac{1}{2}C_2^*\sin 2\psi - \frac{1}{3}C_3^*\cos 3\psi + \frac{1}{4}C_4^*\sin 4\psi + \frac{1}{5}C_5^*\cos 5\psi - - + + \cdots\Big). \end{aligned}\right\} \tag{8}$$

If the new expression (5) is placed in (8, 3) and use is made of the symbol $((\omega))$ defined in Sect. 5, the inertia forces are found to be

$$\left.\begin{aligned} X = \omega^2[&(Q + Q')\cos\psi + Q(A_1^*\sin\psi + A_2^*\cos 2\psi - A_3^*\sin 3\psi - A_4^*\cos 4\psi + \\ &+ A_5^*\sin 5\psi + A_6^*\cos 6\psi - - + + \cdots)] + ((\omega)). \end{aligned}\right\} \tag{9}$$

$$Y = \omega^2 Q'\sin\psi + ((\omega)). \tag{10}$$

For the rolling moment in the offset engine there must be included the angular momentum of the mass $(G's'/l + G'')/g$ of the reciprocating parts, thought of as concentrated at the cross-head or gudgeon-pin,

$$M_z = -B'a - B\frac{\ddot{\chi}}{\lambda} + bQ\frac{\ddot{x}}{r}. \tag{11}$$

On introducing the expressions (8) and (5) and using the combined coefficients

$$F_j = RC_j^* = bQA_j^* \quad (j = 1, 2, \ldots), \tag{12}$$

we find for the rolling moment

$$M_z = \omega^2(-bQ\cos\varphi + F_1\sin\varphi + F_2\cos 2\varphi - F_3\sin 3\varphi - F_4\cos 4\varphi + \\ + F_5\sin 5\varphi + - - + \cdots) + (\omega)_2 - \omega E. \tag{13}$$

If we now compare the results (9), (10), and (13) with the original formulae (5, 1) to (5, 3) we conclude that in general the offset worsens the mass-balancing of the one-cylinder engine except for the transverse force $\mathfrak{Y}$ which is not affected.

Moreover, in the offset one-cylinder engine, balancing of forces of the first order is possible only by making $Q = 0$ and $Q' = 0$, of which the first is practically impossible, as explained in Sect. 6. It is also impossible to eliminate the steady part of the rolling moment since all terms contain bQ. This is because the inertia force of the reciprocating mass no longer is in line with the crank axis but is offset by the amount b. At the most, we can cause one order of the steady rolling moment to vanish (but not the first) by satisfying the condition

$$C_j^*[s'(l - s') - k^2]G' = A_j^* b(s'G' + lG''). \tag{14}$$

Within narrow limits this is possible by suitable shaping of the connecting-rod.

For example, if $G' = G''$ (connecting-rod weight = piston weight), $s' = \frac{1}{2}l$ (centre of gravity of the connecting-rod at its midpoint), and $\lambda = \frac{1}{4}$ and $\beta = \frac{1}{10}$, then $A_3^* = 0{\cdot}007$ and $C_3^* = 0{\cdot}025$ so that for $j = 3$ equation (14) becomes

$$\frac{s'(l - s') - k^2}{l^2} = 0{\cdot}04. \tag{15}$$

This value lies in the range of possible construction. [In normal connecting-rods the quotient $\{s'(l - s') - k^2\}/l^2$ lies between 0 and 0·1.] On the other hand, for $j = 2$ the value 1·5 is obtained which is quite impossible to achieve.

In multi-cylinder engines conditions are somewhat better than in the one-cylinder engine. The tabulation in Sect. 10 indicating the unbalanced orders of forces and moments may be easily extended to two- and four-stroke offset engines by using for the longitudinal forces not only the black circles but also the white, and similarly, for the rolling moments not only the white but also the black. Thus the same orders are unbalanced for longitudinal forces as for rolling moments. Apart from the magnitudes of the unbalanced forces and moments, the offset alters nothing in the order of the balancing of the two-stroke engine with even number of cylinders, and the same is true of the four-stroke engine whose cylinder number is divisible by four. On the other hand, balancing of all other engines is worsened.

by the offset. In respect of the longitudinal forces (Sect. 10), with increased offset the superiority of the two-stroke engine with odd number of cylinders and of the four-stroke with the number of cylinders odd or at least not divisible by four decreases. Similarly, in respect of the rolling moment the superiority of the two-stroke engine with an even cylinder number and of the four-stroke with a cylinder number divisible by four decreases with increasing offset. The magnitude of the unbalanced force is always increased by the offset (since $A_j^* > A_j$). With the multi-cylinder engine also, by satisfying conditions (14), we can eliminate one order of the rolling moment under some conditions.

15. Rotating Balance-weights.—In Sect. 6 it was shown that in the one-cylinder engine the whole transverse force, or the longitudinal force of the first order, can be eliminated, or both forces reduced by arranging counterweights on the cranks according to (6, 3) (or placing them symmetrically with respect to both webs). In the multi-cylinder engine also, where the total transverse and longitudinal forces of first order vanish, such counterweights are sometimes desirable, on the one hand to render harmless the yawing and pitching moments of first order (Sect. 7 to 9), and on the other to reduce the stresses in the individual cranks due to first-order inertia forces.

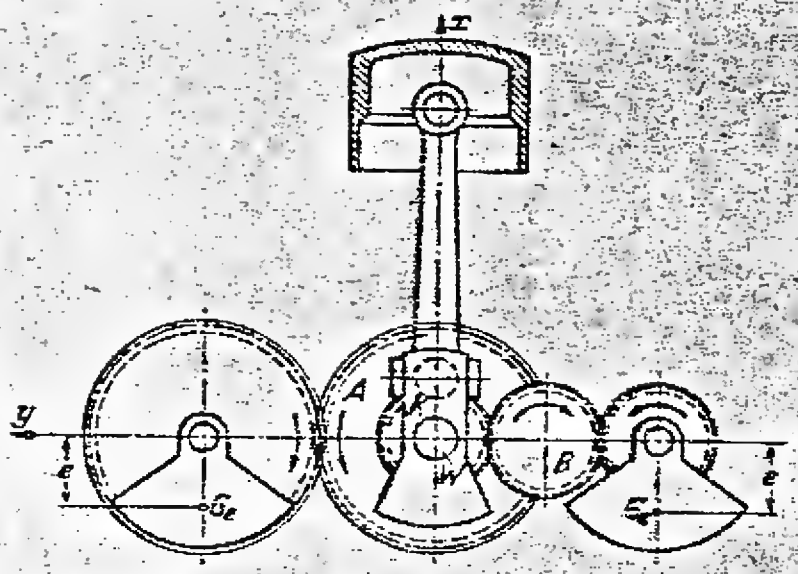


Fig. 30.—Balance-weights for eliminating the first-order longitudinal force.

There is yet another way of eliminating the forces and moments of first order: by use of rotating eccentric balance-weights.[1] This method is mainly used when the complete elimination of all forces or moments of the first order is required; it consists first in employing counterweights on the crank webs to make the reduced moment $Q' = 0$ (6, 1), and then in eliminating Q by means of rotating eccentric weights. We consider this first in connection with the one-cylinder engine, referring to (5, 1).

In order to cancel the first-order component of the longitudinal force

$$X_1 = Q'\omega^2 \cos(\psi + \omega \sin\psi) \quad \text{where} \quad Q = \frac{r}{g}\left(\frac{s}{l} G' + G''\right) \tag{1}$$

which remains after $Q' = 0$, without at the same time producing another unbalanced transverse force, the mechanism of Fig. 30 can be used. This

[1] E. Marquard, "Über den Ausgleich von Beschleunigungskräften zweiter Ordnung durch Drehmassen", *Motorwagen* 29 (1926), p. 328; G. Naske, "Neuere Ölmotorwagen", *Zeit. VDI* 72 (1928), p. 1610.

consists essentially of two equal rotating weights G_e with eccentricity e, coupled by the gears A, B to the crankshaft so that they rotate in opposite directions at the same speed as the engine $\omega_e = \mp\omega$. The eccentrics must both be in the position $\psi_{e0} = 180°$ when the crank is at upper dead centre, $\psi = 0$, as shown in Fig. 30. In order to show that this mechanism does what is expected of it and to find the necessary value of the product eG_e, we must calculate the amounts of the inertia forces due to the eccentric weights. By the method used in (3, 3) we find

$$X_e = \frac{eG_e}{g}[\omega^2 \cos(180° + \psi) + (-\omega)^2 \cos(180° - \psi) + \omega \sin(180° + \psi) + (-\omega) \sin(180° - \psi)]$$

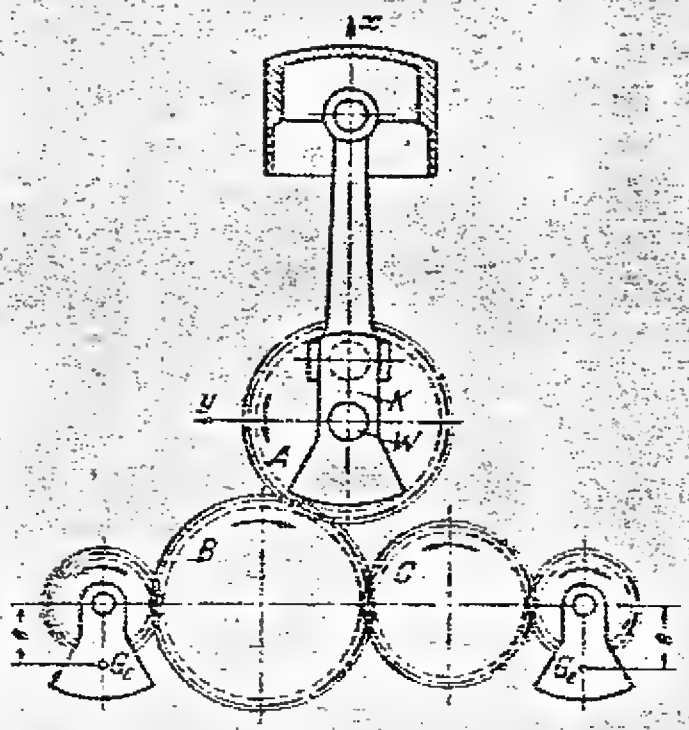


Fig. 31.—Balance-weights for eliminating the second-order longitudinal force.

or

$$X_e = -2\frac{eG_e}{g}(\omega^2 \cos\psi + \omega \sin\psi) \qquad (2)$$

and also (on account of symmetry)

$$Y_e = 0.$$

It follows that for all crank positions the sum $X_1 + X_e = 0$ (without affecting the transverse forces) provided we satisfy the condition

$$eG_e = \frac{1}{2}rG^* \quad \text{where} \quad G^* = \frac{s}{l}G' - G''. \qquad (3)$$

This is possible but requires relatively large weights G_e of the same order as G^*.

Obviously, this scheme can be used to eliminate any particular longitudinal force of higher order; it is more suitable for the higher orders since smaller eccentric weights can be used. For example, if we wish to eliminate the longitudinal force of second order, given by

$$X_2 = A_2 Q\left(\omega^2 \cos 2\psi + \frac{1}{2}\omega \sin 2\psi\right) \qquad (4)$$

the arrangement shown in Fig. 31 (used in engines for rail vehicles) can be used. The eccentric weights G_e are driven in opposite directions with angular velocities $\omega_e = \mp 2\omega$ by means of the gears A, B, C. The zero position is again $\psi_{e0} = 180°$. Then

$$X_e = \frac{eG_e}{g}[(2\omega)^2 \cos(180° + 2\psi) + (-2\omega)^2 \cos(180° - 2\psi) + (2\omega) \sin(180° + 2\psi) + (-2\omega) \sin(180° - 2\psi)]$$

or

$$X_e = -8\frac{eG_e}{g}\left(\omega^2 \cos 2\psi + \frac{1}{2}\omega \sin 2\psi\right) \quad \text{and} \quad Y_e = 0, \qquad (5)$$

so that always $\Sigma_z + \Sigma_r = 0$ provided

$$eG_e = \frac{1}{8} A_2 rG^*. \tag{6}$$

Since A_2 is approximately equal to λ (4, 6b), this condition (6) can be satisfied with relatively small eccentric weights and with small eccentricities e. Fig. 32 shows how balancing of both first and second orders can be achieved simultaneously.

In similar fashion the longitudinal force of fourth order can be balanced by means of two eccentric weights G_e rotating in opposite directions at the rate 4ω. However, these must be so arranged that they reach the zero position $\varphi_{e0} = 0$ simultaneously with the crank (as shown in 5, 1). Then

$$eG_e = \frac{1}{32} A_4 rG^*. \tag{7}$$

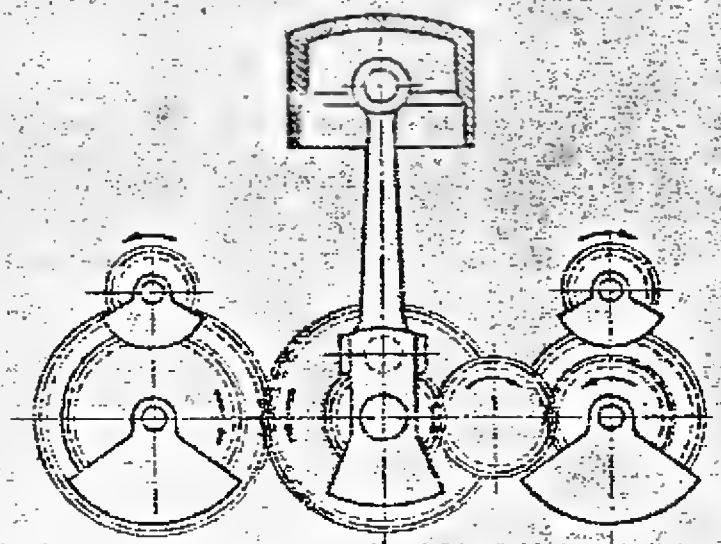


Fig. 32.—Balance-weights for eliminating the first- and second-order longitudinal forces.

To eliminate the longitudinal force of jth order in the one-cylinder engine we have to satisfy the conditions

$$\left.\begin{aligned} eG_e &= \frac{1}{2j^2} A_j rG^*, \quad \omega_e = \pm j\omega, \\ \varphi_{e0} &= j \times 90^\circ, \ (j = 2, 4, 6, \ldots). \end{aligned}\right\} \tag{8}$$

In similar fashion the two-cylinder two-stroke engine (Sect. 7a) can be improved by the use of two eccentric weights whose centre of gravity moves in the plane of symmetry ($z = 0$) of both cylinders (in order to produce no moment) and for which the condition, analogous to (8), holds:

$$eG_e = \frac{1}{j^2} A_j rG^*, \quad \omega_e = \pm j\omega, \quad \varphi_{e0} = j \times 90^\circ. \tag{9}$$

Another possibility is to use four eccentric weights arranged in pairs symmetrically with respect to the (x, y)-plane so that this plane and the (x, z)-plane are planes of kinematical symmetry; this leads to equations (8). (It is easy to show both here and below that with respect to the zero position of the eccentric φ_{e0} corresponding to $\varphi = 0$ of the crank, it makes no difference on which crank φ is measured; $\varphi_{e0} = 180^\circ$ simply means that both cranks pass through the position $\varphi = 180^\circ$ at the same instant as the eccentrics. Similarly, $\varphi_{e0} = 0$ means that the cranks and the eccentrics go through $\varphi = 0$ together.)

The two-cylinder four-stroke engine (Sect. 7b) is equivalent to a one-

cylinder engine with doubled weights, so far as the rotating eccentric weights are concerned.

Similarly, for the eccentrics of the three-cylinder engine

$$eG_e = \left[\frac{1}{2}\right]\frac{3}{2\pi} A_j r G^*, \quad \omega_e = \pm j\omega, \quad \psi_{e0} = j \times 90°. \tag{10}$$

For the four-cylinder engine

$$eG_e = \left[\frac{1}{2}\right]\frac{2}{j^2} A_j r G^*, \quad \omega_e = \pm j\omega, \quad \psi_{e0} = j \times 90°. \tag{11}$$

In general, for n cylinders

$$eG_e = \left[\frac{1}{2}\right]\frac{n}{2j^2} A_j r G^*, \quad \omega_e = \pm j\omega, \quad \psi_{e0} = j \times 90°. \tag{12}$$

The factor in square brackets $[\frac{1}{2}]$ is used for four eccentrics but is omitted for two. Whether the two- or four-stroke cycle is used is unimportant, but regular crank angles are always assumed.

In the same way as the inertia forces, the inertia moments of any particular order can be cancelled by use of rotating balance-weights. We ignore the rolling moment M_x whose terms of first and higher order can be eliminated by use of properly shaped connecting-rods (6, 9), and whose zero-order term can only be cancelled by a relatively large counter-rotating mass which can be calculated by the second of equations (6, 8). By using crank counterweighus we make $Q' = 0$ and, consequently, $M_x = 0$ ((7, 2), (8, 2), (9, 3) to (9, 5) and (11, 2)); then by use of rotating balance-weights we can ensure that one term of M_y, of any desired order, vanishes.

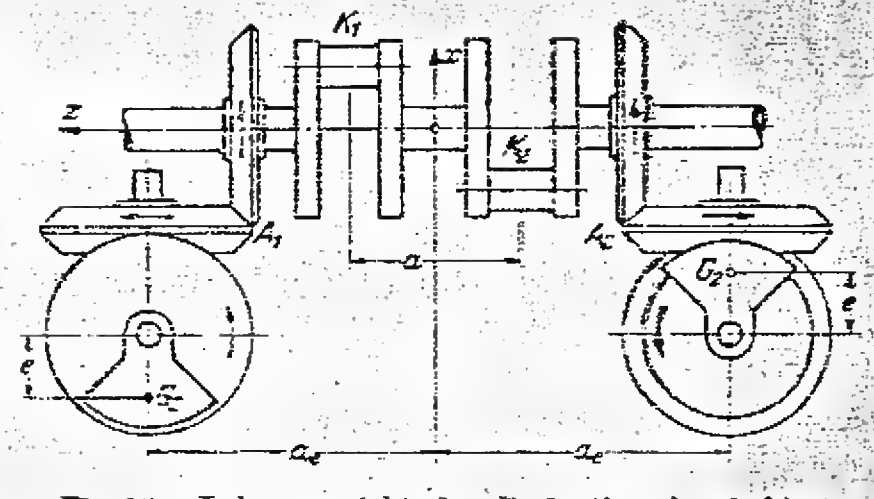


Fig. 33.—Balance-weights for eliminating the pitching moment in the two-cylinder engine.

This will first be demonstrated for the pitching moment M_y in the two-cylinder two-stroke engine of Sect. 7, for which

$$M_y = a\mathfrak{P}(\omega^2 \cos\psi + \dot{\omega}\sin\psi). \tag{13}$$

In the mechanism shown in Fig. 33 the bevel gears A_1 and A_2 turn (all in the same direction) four eccentric weights G_e which are arranged in pairs, symmetric with respect to the (x, z)-plane and have the eccentricity e. They rotate at the same angular velocity as the engine. The z-coordinates of the

axes of the rotating weights are $-a_e$. When the crank K_1 is at $\psi_{(1)} = 0$ the other crank is at $\psi_{(2)} = 180°$, the eccentric weights on the side of K_1 are at $\psi_{e1} = 180°$ and the others at $\psi_{e2} = 0°$. The angles ψ_e are considered positive when the rotation ω_e would cause a right-hand screw to move along the negative y-axis (towards the observer in Fig. 33). The eccentric weights then give rise to inertia moments as follows:

$$M_{ex} = 0, \qquad M_{ez} = 0$$

and

$$M_{ey} = [2]\, a_e \frac{eG_e}{g} [\omega^2 \cos(180° + \psi) + \dot{\omega} \sin(180° + \psi)] - [2]\, a_e \frac{eG_e}{g} (\omega^2 \cos\psi + \dot{\omega} \sin\psi).$$

Therefore

$$M_{ey} = -[2] 2 a_e \frac{eG_e}{g} (\omega^2 \cos\psi + \dot{\omega} \sin\psi). \tag{14}$$

Thus, M_y is entirely eliminated if we satisfy the conditions

$$a_e eG_e = \left[\frac{1}{2}\right] \frac{1}{2}\, arG^*. \tag{15}$$

The factors in square brackets [2] or [½] are omitted if only two eccentric weights, which must then be in the (x, y)-plane, are used.

Exactly the same procedure is possible with the three-cylinder engine provided, as shown by (8, 2), we satisfy the conditions

$$a_e eG_e = \left[\frac{1}{2}\right] \frac{\sqrt{3}}{2}\, arG^*, \quad \psi_{e1} = 270°, \quad \psi_{e2} = 90°. \tag{16}$$

The zero position $\psi_{(1)} = 0$ refers to the centre crank.

In the four-cylinder two-stroke engine the pitching moment of first order vanishes for each of the three possible crank and timing arrangements (9, 3) to (9, 5), provided

$$\left.\begin{aligned}
&\text{(a)}\ a_e eG_e = \left[\frac{1}{2}\right] \sqrt{2}\, arG^*, && \psi_{e1} = 225°, && \psi_{e2} = 45°,\\
&\text{(b)}\ a_e eG_e = \left[\frac{1}{2}\right] \frac{\sqrt{10}}{2}\, arG^*, && \psi_{e1} = 198{\cdot}4°, && \psi_{e2} = 18{\cdot}4°,\\
&\text{(c)}\ a_e eG_e = \left[\frac{1}{2}\right] \frac{\sqrt{2}}{2}\, arG^*, && \psi_{e1} = 225°, && \psi_{e2} = 45°.
\end{aligned}\right\} \tag{17}$$

In general, with the n-cylinder two-stroke engine, we must satisfy

$$a_e eG_e = \left[\frac{1}{2}\right] \frac{\varkappa_1}{2}\, arG^*, \quad \psi_{e1} = 180° + \beta_1, \quad \psi_{e2} = \beta_1, \tag{18}$$

where $\varkappa_1$ and β_1 are the numbers tabulated in Sect. 11. The zero positions of the eccentric weights ψ_{e1} and ψ_{e2} are calculated from the zero position $\psi_{(1)} = 0$ of crank 1.

Moreover, each pitching moment of higher order which is not automatically zero can be balanced in this fashion. For balancing the pitching moment of jth order in the n-cylinder engine the conditions are

$$a_e e G_e = \left[\frac{1}{2}\right] \frac{\alpha_j}{2j^2} A_j a r G^*, \quad \omega_e = +j\omega, \quad \nu_{e1} = 180^\circ + \beta_j, \quad \nu_{e2} = \beta_j, \tag{19}$$

in which the numbers α_j and β_j are given in Sect. 11. The gear ratio must be such that the eccentric weights rotate with angular velocity j times that of the engine.

For example, in a six-cylinder two-stroke engine with firing order 1 5 3 4 2 6 and $\lambda = \frac{1}{4}$, the second-order pitching moment is balanced provided

$$a_e e G_e = \left[\frac{1}{2}\right] 0{\cdot}110 a r G^*, \quad \omega_e = +2\omega, \quad \nu_{e1} = 210^\circ, \quad \nu_{e2} = 30^\circ.$$

These conditions are easily satisfied constructionally.

For the four-stroke engine with odd number of cylinders the same conditions hold as for the two-stroke engine, whereas with an even number of cylinders the pitching moments are balanced in quite another fashion (Sect. 11b).

§ 4. V-, Multi-row, and Radial Engines with Centrally Attached Connecting-rods

16. The V-engine.—The V-engine consists of two standard reciprocating mechanisms lying in the (x, y)-plane and making the V-angle δ (Fig. 34). There is a single crank, common to both mechanisms; it is first assumed that both mechanisms are identical and that both connecting-rods are attached directly to the crankpin A, one being displaced slightly with respect to the other along the crank axis. Later (§ 5) we shall discuss the other common case in which one of the connecting-rods (the articulated rod) is attached to the other rod (the master rod) at a point other than A.

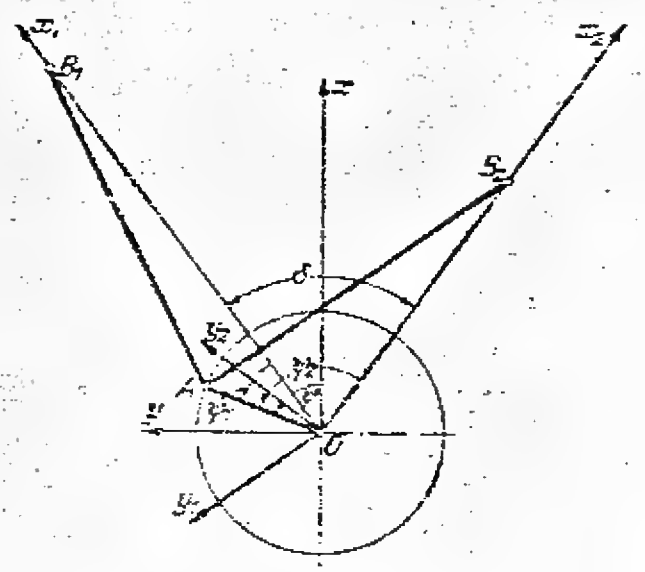


Fig. 34.—Two-row or V-engine.

To the first reciprocating mechanism we assign a coordinate system (x_1, y_1), to the other system (x_2, y_2), and to the whole engine (x, y), in which the x-axis bisects the angle δ. The rotation of the crank with respect to each of the axes x_1, x_2, x is ψ_1, ψ_2, ψ, respectively, so that

$$\psi_1 = \psi - \frac{\delta}{2}, \quad \psi_2 = \psi + \frac{\delta}{2}. \tag{1}$$

The inertia forces X_1, Y_1 of the first mechanism in the (x_1, y_1)-system and X_2, Y_2 of the second in the (x_2, y_2)-system are given by (5, 1) and (5, 2):

$$\left.\begin{aligned} X_i &= \omega^2[(Q+Q'')\cos\psi_i + Q(A_2\cos 2\psi_i - A_4\cos 4\psi_i - \cdots)] + ((\omega)),\\ Y_i &= \omega^2 Q''\sin\psi_i + ((\omega)). \end{aligned}\right\} \quad (i=1,2). \tag{2}$$

Here, as before, Q is the reduced first moment (3, 2) whereas, instead of Q', the quantity

$$Q'' = \frac{1}{g}\left[\frac{1}{2}sG + r\left(1-\frac{\varepsilon}{l}\right)G'\right] \tag{3}$$

must be used since now only half the crank mass is assigned to each mechanism.

The forces X_i, Y_i have resultants in the x- and y-directions:

$$\left.\begin{aligned} X &= (X_1 + X_2)\cos\frac{\delta}{2} - (Y_1 - Y_2)\sin\frac{\delta}{2},\\ Y &= (X_1 - X_2)\sin\frac{\delta}{2} + (Y_1 + Y_2)\cos\frac{\delta}{2}. \end{aligned}\right\} \tag{4}$$

On introducing equations (2) and observing that

$$\cos j\psi_1 + \cos j\psi_2 = 2\cos j\frac{\delta}{2}\cos j\psi, \qquad \cos j\psi_1 - \cos j\psi_2 = 2\sin j\frac{\delta}{2}\sin j\psi,$$

$$\sin j\psi_1 + \sin j\psi_2 = 2\cos j\frac{\delta}{2}\sin j\psi, \qquad \sin j\psi_1 - \sin j\psi_2 = -2\sin j\frac{\delta}{2}\cos j\psi$$

we get the longitudinal and transverse forces in the V-engine:

$$\left.\begin{aligned} X &= \omega^2\left\{[Q(1+\cos\delta)+2Q'']\cos\psi - 2Q\cos\frac{\delta}{2}(A_2\cos\delta\cos 2\psi - A_4\cos 2\delta\cos 4\psi + - \cdots)\right\} + ((\omega)),\\ Y &= \omega^2\left\{[Q(1-\cos\delta)+2Q'']\sin\psi - 2Q\sin\frac{\delta}{2}(A_2\sin\delta\sin 2\psi - A_4\sin 2\delta\sin 4\psi + - \cdots)\right\} + ((\omega)). \end{aligned}\right\} \tag{5}$$

[In writing the symbol $((\omega))$ it is understood that only the quantities $\cos j\psi$ and $\sin j\psi$ are to be changed, not the constant terms $\cos\frac{1}{2}j\delta$ and $\sin\frac{1}{2}j\delta$.]

Similarly, from (5, 3) with the reduced moment of inertia R (3, 6) and with

$$R'' = \frac{1}{g}\left[\frac{1}{2}k^2G + r^2\left(1-\frac{\varepsilon}{l}\right)G'\right] \tag{6}$$

the rolling moment of the V-engine is obtained:

$$M_x = 2\omega^2 R\left(C_1\cos\frac{\delta}{2}\sin\psi - C_3\cos\frac{3\delta}{2}\sin 3\psi + - \cdots\right) + ((\omega)) - 2\omega^2 R''. \tag{7}$$

Next we consider the possibilities of balancing. Since it is not possible, for reasons of construction, to make both $Q = 0$ and $Q'' = 0$ simultaneously, complete balancing cannot be achieved. Therefore, we shall limit ourselves

to forces of the first order, as being the most important. According to (5), these vanish provided

$$Q(1+\cos\delta)+2Q''=0 \quad \text{and} \quad Q(1-\cos\delta)+2Q''=0.$$

By addition and subtraction we obtain the following necessary and sufficient conditions for balancing first-order forces:

$$Q+2Q''=0 \quad \text{and} \quad \cos\delta=0, \quad \text{so that} \quad \delta=90°. \tag{8}$$

The first condition can be satisfied by using counterweights on backward extensions of the crank webs, so that with s negative,

$$sG+r\left[\left(2-\frac{s'}{l}\right)G'+G''\right]=0. \tag{9}$$

This condition is rather awkward because it requires large counterweights. For example, if $s'=\frac{1}{2}l$ and if $s=-r$, the counterweight (including the crank itself) amounts to

$$G=\frac{3}{2}G'+G''.$$

The second of the conditions (8) requires the cylinders to be at right angles to each other, resulting in a very wide engine.

If the conditions (8) are not satisfied the unbalanced force of first order can be represented by four revolving vectors of lengths

$$\omega^2(Q+2Q''), \quad \omega^2 Q\cos\delta, \quad \omega(Q+2Q''), \quad \omega Q\cos\delta,$$

whose directions are shown in Fig. 35 (for $\cos\delta>0$ and $\omega>0$). These directions are obtained from the instantaneous crank rotation ψ by reflections, partly in the x-axis, partly in the median M. These vectors have the x- and y-components

$$\omega^2[Q(1+\cos\delta)+2Q'']\cos\psi+\omega[Q(1+\cos\delta)+2Q'']\sin\psi$$

and

$$\omega^2[Q(1-\cos\delta)+2Q'']\sin\psi-\omega[Q(1-\cos\delta)+2Q'']\cos\psi$$

in agreement with (5).

If mechanisms like those of Sect. 15 are not considered (although they can be used very easily in this situation and are easily calculated) the inertia forces of higher order in V-engines can generally not be balanced because $Q\neq 0$. However, cases in which the V-angle δ is so chosen that either $\cos\frac{1}{2}j\delta$ or $\sin\frac{1}{2}j\delta$ vanishes are exceptions. For example, if $\delta=90°$, in the longitudinal force X the terms of order 2, 6, 10 . . . vanish, as do those of order 4, 8, 12 . . . in the transverse force Y. If $\delta=45°$ the orders 4, 12 . . . in X and 8, 16 . . . in Y are missing. The unbalanced steady forces of higher order can be represented as vectors whose ends trace out ellipses with eccentric anomalies equal to $j\psi-(j-2)90°$. The principal axes of

the ellipses coincide with the x- and y-axes and have the semi-diameters $2\omega^2 Q A_1 \cos\frac{1}{2}\delta \cos\frac{1}{2}j\delta$ and $2\omega^2 Q A_1 \sin\frac{1}{2}\delta \sin\frac{1}{2}j\delta$. The higher-order unsteady forces behave similarly.

For balancing the rolling moment M_z the statement made in Sect. 6 still holds: because $R' \neq 0$ the unsteady moment of zero order $-2\omega R''$ cannot be balanced (except by use of eccentric masses as in Sect. 15). The rest of the rolling moment, especially the whole steady part, vanishes, providing the connecting-rods satisfy the condition $R = 0$ (i.e. $k'^2 = s'(l - s')$).

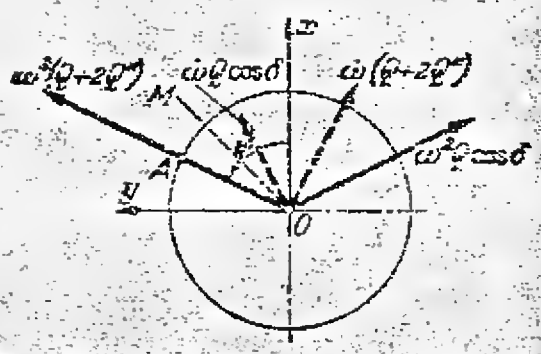


Fig. 35.—Unbalanced inertia forces of the first order in the V-engine.

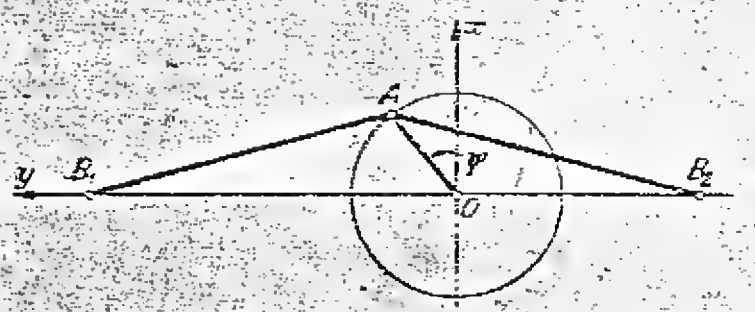


Fig. 36.—Two-cylinder opposed engine.

17. Opposed Engines.—There is a special type of V-engine in which all forces and moments of higher order are missing, although at the expense of first-order balance. This is the "opposed engine" with V-angle of 180° or the two-cylinder radial engine (Fig. 36). From (16, 5) and (16, 7)

$$\left.\begin{aligned} X &= 2\omega^2 Q' \cos\varphi + ((\omega)), \\ Y &= 2\omega^2 (Q + Q') \sin\varphi + ((\omega)), \\ M_z &= -2\omega R'. \end{aligned}\right\} \tag{1}$$

The first-order forces in this engine cannot be entirely removed except by use of rotating balance-weights, although they can be reduced by means of counterweights on the crank (6, 3).

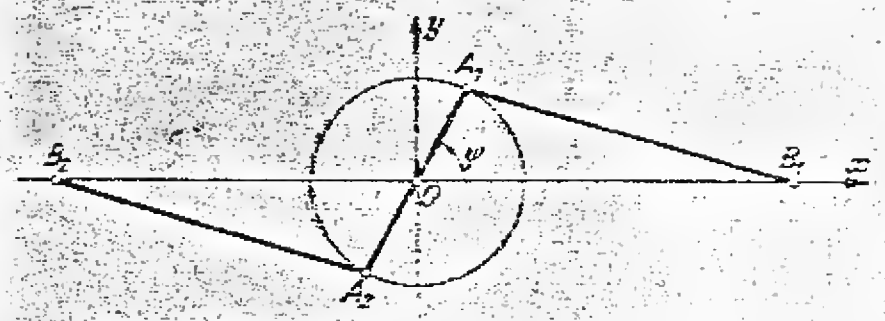


Fig. 37.—Two-cylinder two-crank opposed engine.

This single-crank two-cylinder engine must be distinguished from the double-crank opposed two-cylinder engine with 180° crank offset (Fig. 37), for which

$$\left.\begin{aligned} X &= 0, \\ Y &= 0, \\ M_z &= 2\omega^2 R (C_2 \sin 2\varphi - C_4 \sin 4\varphi + \cdots) + ((\omega)) - 2\omega R'. \end{aligned}\right\} \tag{2}$$

The forces are automatically balanced because of symmetry, whereas the rolling moments combine.

Finally we consider the double-piston engine, available in two types (Figs. 38 and 39), with similarly or with oppositely rotating cranks that are connected together by some mechanism. In the first type (Fig. 38) all forces balance while the rolling moments of the two mechanisms combine and must be added to the moment $-aY_1$ of the two transverse forces, where a is the separation of the two cranks and Y_1 is the transverse force of a single reciprocating mechanism. Then

$$\left.\begin{aligned} X &= 0, \\ Y &= 0, \\ M_z &= \omega^2[-aQ'\sin\psi + 2R(C_1\sin\psi - C_2\sin 3\psi + \cdots)] + ((\omega)) - 2\omega R'. \end{aligned}\right| \tag{3}$$

In the second type (Fig. 39) both X and M_z vanish while the transverse forces combine

$$\left.\begin{aligned} X &= 0, \\ Y &= 2\omega^2 Q'\sin\psi + ((\omega)), \\ M_z &= 0. \end{aligned}\right| \tag{4}$$

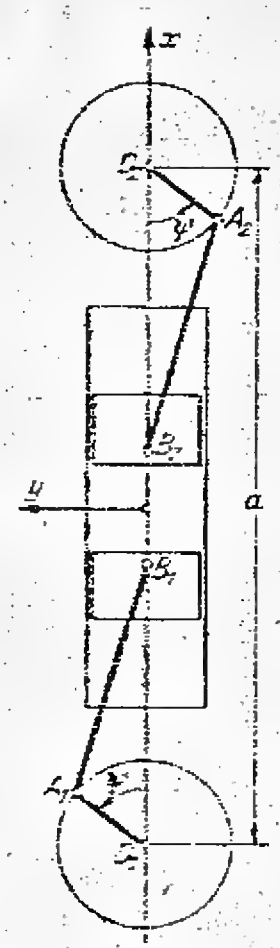


Fig. 38.—Double-piston engine.

Since we can make $Q' = 0$ with no difficulty by means of crank counterweights (6, 1), the second type is a completely balanceable engine. In the first type, at least the steady part of the rolling moment can be eliminated by use of suitably shaped connecting-rods to make $R = 0$ (6, 9) in addition to placing counterweights on the cranks. In any case, with respect to balancing, the second type is definitely superior to the first, which was formerly the only one used.

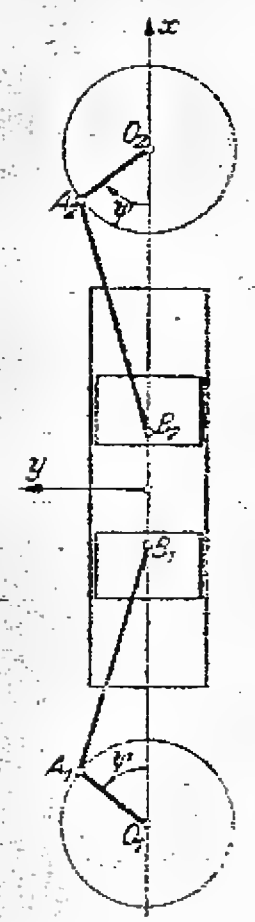


Fig. 39.—Double-piston engine.

18. The Multi-row Engine.—If a third reciprocating mechanism is added to the V-engine symmetrically between the original mechanisms, the result is the three-cylinder (W- or fan-type) engine (Fig. 40). To find the inertia forces and moments with identical pistons and rods, the rods being coupled to the crank itself and not articulated, it is only necessary to add the quantities defined by (5, 1) to (5, 3) for the centre mechanism to expressions (16, 5) and (16, 7). Also, δ is replaced by 2δ and the reduced first moments Q' and Q'' by

$$Q''' = \frac{l}{s}\left[\frac{1}{3}\delta G + r\left(1 - \frac{s}{l}\right)G'\right] \tag{1}$$

and finally, R' and R'' by

$$R''' = \frac{1}{g}\left[\frac{1}{3}\,\varepsilon^2 G + r^2\left(1 - \frac{s}{l}\right)G'\right], \tag{2}$$

since the mass of the crank must be divided among the three mechanisms. The result is

$$\left.\begin{aligned} X &= \omega^2\{[Q(2 + \cos 2\delta) + 3Q'''] \cos\psi + Q[A_2(1 + 2\cos\delta\cos 2\delta)\cos 2\psi - \\ &\qquad - A_4(1 + 2\cos\delta\cos 4\delta)\cos 4\psi + - \cdots]\} + ((\omega)), \\ Y &= \omega^2\{[Q(1 - \cos 2\delta) + 3Q'''] \sin\psi + 2Q\sin\delta\,(A_2\sin 2\delta\sin 2\psi - \\ &\qquad - A_4\sin 4\delta\sin 4\psi + - \cdots)\} + ((\omega)), \\ M_z &= \omega^2 R[C_1(1 + 2\cos\delta)\sin\psi - C_3(1 + 2\cos 3\delta)\sin 3\psi + - \cdots] + ((\omega)) - 3\omega R'''. \end{aligned}\right\} \tag{3}$$

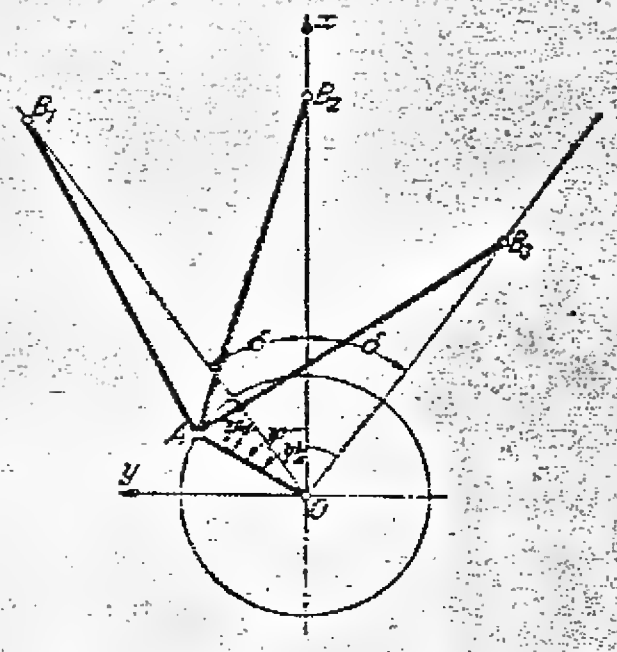


Fig. 40.—Three-row or W-engine.

For the forces of first order to vanish the two following conditions must be satisfied simultaneously:

$$Q(2 + \cos 2\delta) + 3Q''' = 0 \quad \text{and} \quad Q(1 - \cos 2\delta) + 3Q''' = 0.$$

By addition and subtraction, the conditions for balancing the forces of first order are found to be

$$Q + 2Q''' = 0 \quad \text{and} \quad \cos 2\delta = -\frac{1}{2}, \quad \text{so that } \delta = 60° \text{ (or } 120°). \tag{4}$$

In addition to a large counterweight on the crank which (s being negative) satisfies the equation

$$sG + \frac{3}{2}r\left[\left(2 - \frac{s'}{l}\right)G' + G''\right] = 0, \tag{5}$$

cylinder angles $\delta = 60°$ are required. The case $\delta = 120°$ is discussed under radial engines in Sect. 19.

If the conditions (4) are not satisfied, the unbalanced force of first order can be represented by four rotating vectors of lengths

$$\frac{3}{2}\omega^2(Q+2Q'''), \quad \omega^2 Q\left(\frac{1}{2}+\cos 2\delta\right), \quad \frac{3}{2}\omega(Q+2Q'''), \quad \omega Q\left(\frac{1}{2}+\cos 2\delta\right),$$

with exactly the same directions as in the V-engine (Fig. 35 of Sect. 16). Moreover, for the forces of higher order as well as the rolling moment M_z everything is as given in Sect. 16. The second-order terms in X vanish for $\delta = 152{\cdot}3°$, the fourth-order for $\delta = 31{\cdot}5°$, the sixth-order for $\delta = 20{\cdot}4°$. The second-order Y terms vanish for $\delta = 90°$, the fourth-order for $\delta = 45°$, etc.

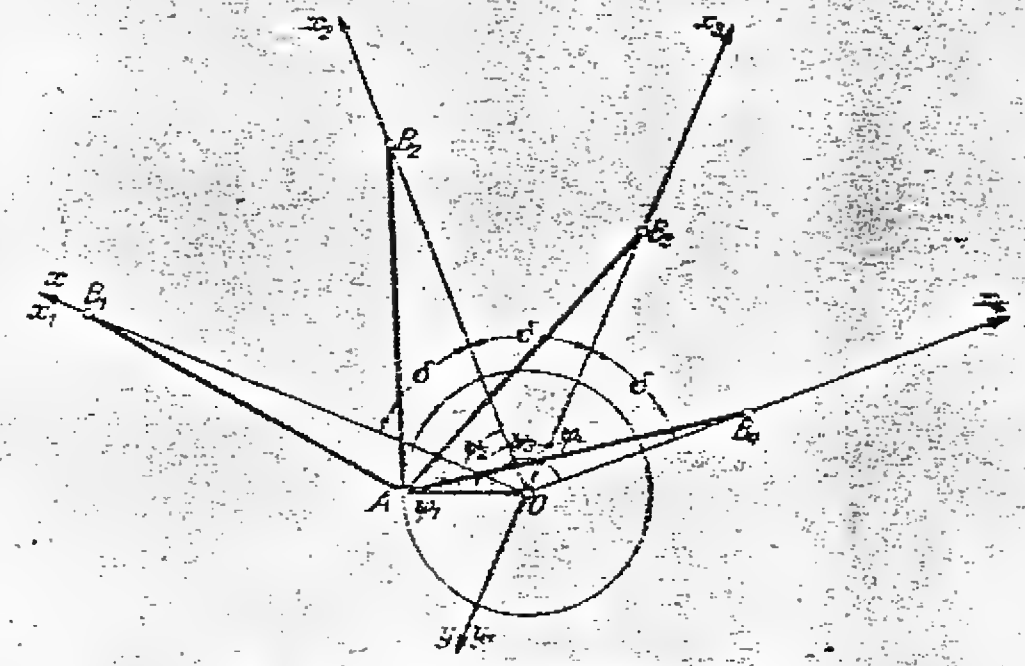


Fig. 41.—Four-cylinder fan-type engine.

Conditions (4), as well as (16, 8) are special cases of a general rule, according to which an engine with p cylinders arranged fanwise with equal angles and having identical pistons and rods (not articulated), is balanced for first-order forces, provided

$$Q + 2Q^{(p)} = 0 \quad \text{and} \quad \delta = \frac{180°}{p}, \tag{6}$$

where

$$Q^{(p)} = \frac{1}{\varrho}\left[\frac{1}{p}sG + r\left(1-\frac{\varepsilon}{l}\right)G'\right]. \tag{7}$$

Therefore, equation (6) can be written explicitly

$$sG + \frac{p}{2}r\left[\left(2-\frac{\varepsilon}{l}\right)G' + G'''\right] = 0. \tag{8}$$

This demands a large crank counterweight.

In order to prove this result we use Fig. 41, which represents the case $p = 4$, letting the (x, y)-axes coincide with (x_1, y_1). Since

$$\delta_i = (i-1)\delta \quad \text{and} \quad \varphi_i = \varphi + \delta_i \qquad (i = 1, 2, \ldots, p), \tag{9}$$

we get

$$X = \sum_{i=1}^{p} X_i \cos\delta_i + \sum_{i=1}^{p} Y_i \sin\delta_i, \quad Y = -\sum_{i=1}^{p} X_i \sin\delta_i + \sum_{i=1}^{p} Y_i \cos\delta_i. \tag{10}$$

Introducing expressions for X_i and Y_i from (5, 1) and (5, 2), writing $Q^{(p)}$ instead of Q', and limiting ourselves to first-order terms, we obtain

$$X = \omega^2[(Q + Q^{(p)})\Sigma\cos\psi_i\cos\delta_i + Q^{(p)}\Sigma\sin\psi_i\sin\delta_i] + ((\omega)).$$

Rewriting the coefficients of Q and $Q^{(p)}$ with use of the second equation (9), we find

$$\Sigma\cos\psi_i\cos\delta_i = \frac{1}{2}\Sigma\cos(\psi_i+\delta_i) + \frac{1}{2}\Sigma\cos(\psi_i-\delta_i) = \frac{1}{2}\Sigma\cos(\psi+2\delta_i) + \frac{1}{2}p\cos\psi,$$

$$\Sigma\cos\psi_i\cos\delta_i + \Sigma\sin\psi_i\sin\delta_i = \Sigma\cos(\psi_i-\delta_i) = p\cos\psi.$$

The same can be done for the expression $((\dot\omega))$ so that

$$\left.\begin{aligned} X &= \frac{1}{2}\omega^2[p(Q+2Q^{(p)})\cos\psi + Q\Sigma\cos(\psi+2\delta_i)] + ((\omega)) \\ \text{and by analogy} \\ Y &= \frac{1}{2}\omega^2[p(Q+2Q^{(p)})\sin\psi - Q\Sigma\sin(\psi+2\delta_i)] + ((\omega)). \end{aligned}\right\} \tag{11}$$

The first of the terms in each of the square brackets vanishes as the result of the first condition (6), while the second terms vanish as the result of the second condition (6), as can be seen by applying the summation formula (10, 4a):

$$\Sigma\cos(\psi+2\delta_i) \equiv \Sigma\cos\left[\psi+(i-1)\frac{2\pi}{p}\right] = 0, \quad \Sigma\sin(\psi+2\delta_i) \equiv \Sigma\sin\left[\psi+(i-1)\frac{2\pi}{p}\right] = 0.$$

Thus the relation is proved.

It can be seen that these summations also vanish when $\delta = 2\pi/p$ (the radial engine), except in the case $p = 2$ (Sect. 17).

18. The Regular Radial Engine.—The radial engine normally consists of p identical reciprocating mechanisms making equal angles $\delta = 2\pi/p$ with one another (Fig. 42). In the following, we omit the two-cylinder radial (opposed) engine already discussed in Sect. 17. We first consider the case in which all connecting-rods are connected to the crankpin A (i.e. are not articulated). Then we can use equations (18, 7), (18, 9), and (18, 10) with the same symbols, observing the final remarks of Sect. 18 and introducing the terms of higher order into (18, 11), to obtain

$$\left.\begin{aligned} X &= \omega^2\left[\frac{p}{2}(Q+2Q^{(p)})\cos\psi + Q(A_p\Sigma\cos 2\psi_i\cos\delta_i - A_{2p}\Sigma\cos 4\psi_i\cos\delta_i + -\cdots)\right] + ((\omega)), \\ Y &= \omega^2\left[\frac{p}{2}(Q+2Q^{(p)})\sin\psi - Q(A_p\Sigma\cos 2\psi_i\sin\delta_i - A_{2p}\Sigma\cos 4\psi_i\sin\delta_i + -\cdots)\right] + ((\omega)). \end{aligned}\right\} \tag{1}$$

The summations are for $i = 1$ to p and are easily evaluated. For example, according to (18, 9) for any order j

$$\Sigma \cos j\psi_i \cos \delta_i = \frac{1}{2}\Sigma \cos (j\psi_i + \delta_i) + \frac{1}{2}\Sigma \cos (j\psi_i - \delta_i)$$

$$= \frac{1}{2}\Sigma \cos\left[j\psi + (i-1)(j+1)\frac{2\pi}{p}\right] + \frac{1}{2}\Sigma \cos\left[j\psi + (i-1)(j-1)\frac{2\pi}{p}\right].$$

According to (10, 4a) and (10, 4b) this is zero for $j \pm 1 \neq p, 2p, 3p, \ldots$ and

$$\Sigma \cos j\psi_i \cos \delta_i = \frac{1}{2} p \cos j\psi \quad \text{for} \quad j \pm 1 = p, 2p, 3p \ldots . \tag{2}$$

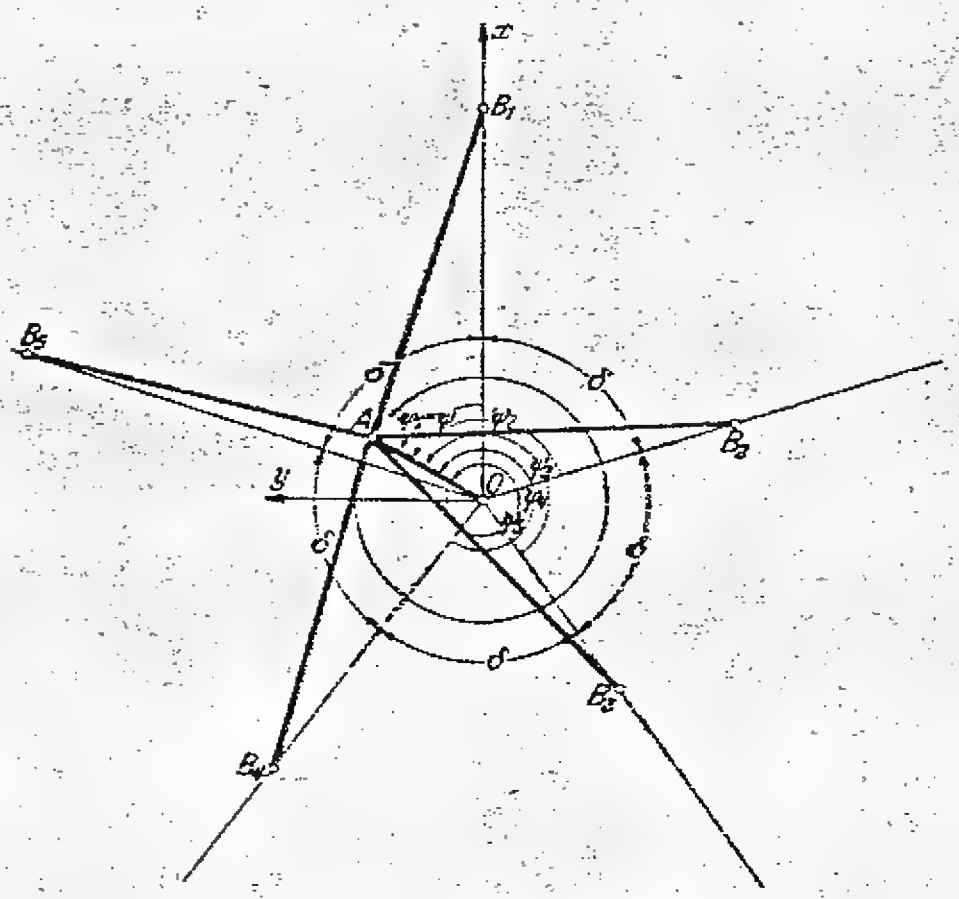


Fig. 42.—Five-cylinder radial engine.

Similarly, the remaining summations are generally zero, except

$$\left.\begin{aligned} \Sigma \cos j\psi_i \sin \delta_i &= \pm \frac{1}{2} p \sin j\psi, \\ \Sigma \sin j\psi_i \cos \delta_i &= \frac{1}{2} p \sin j\psi, \\ \Sigma \sin j\psi_i \sin \delta_i &= \mp \frac{1}{2} p \cos j\psi \end{aligned}\right\} \quad \text{for} \quad j \pm 1 = p, 2p, 3p \ldots . \tag{3}$$

Since the j's in (1) are only even numbers, for an even number of cylinders $p > 2$ all the summations disappear and we have simply

$$\left.\begin{aligned} X^* &= \frac{p}{2}\omega^2 (Q + 2Q^{(x)}) \cos \psi + ((\omega)), \\ Y^* &= \frac{p}{2}\omega^2 (Q + 2Q^{(x)}) \sin \psi + ((\omega)) \end{aligned}\right\} \quad \text{for} \quad p = 4, 6, 8, \ldots . \tag{4}$$

On the other hand, if p is odd there must be added to the first-order terms X^*, Y^* the terms of higher order due to (2) and (3) so that

$$\left.\begin{aligned} X &= X^* \pm \frac{p}{2}\omega^2 Q\left[A_{p-1}\cos(p-1)\psi - A_{p+1}\cos(p+1)\psi - \right.\\ &\quad \left. - A_{3p-1}\cos(3p-1)\psi + A_{3p+1}\cos(3p+1)\psi + - - + \cdots\right] \pm ((\dot\omega)),\\ Y &= Y^* \mp \frac{p}{2}\omega^2 Q\left[A_{p-1}\sin(p-1)\psi + A_{p+1}\sin(p+1)\psi - \right.\\ &\quad \left. - A_{3p-1}\sin(3p-1)\psi - A_{3p+1}\sin(3p+1)\psi + + - - \cdots\right] \mp ((\dot\omega)) \end{aligned}\right\} \text{for } p = \begin{matrix}\{3, 7, 11, \ldots\}\\ \{5, 9, 13, \ldots\}\end{matrix} \tag{5}$$

From (5, 3) the rolling moment is

$$M_z = \omega^2 R(C_1 \Sigma \sin\psi_i - C_3 \Sigma \sin 3\psi_i + - \cdots) + ((\dot\omega)) - p\dot\omega R^{(p)}$$

in which

$$R^{(p)} = \frac{1}{g}\left[\frac{1}{p}k^2 G + r^2\left(1 - \frac{s'}{l}\right)G'\right]. \tag{6}$$

The summations are already evaluated in (10, 4c), giving

$$\left.\begin{aligned} &M_z = -p\dot\omega R^{(p)} \quad \text{for} \quad p = 2, 4, 6, \ldots,\\ &M_z = \mp p\omega^2 R(C_p \sin p\psi - C_{3p}\sin 3p\psi + - \cdots) \mp ((\dot\omega)) - p\dot\omega R^{(p)} \text{ for } p = \begin{matrix}\{3, 7, 11, \ldots\}\\ \{5, 9, 13, \ldots\}\end{matrix} \end{aligned}\right\} \tag{7}$$

The following conclusions can be drawn:

(1) For any number of cylinders p the first-order forces X^*, Y^* can be balanced by means of a large crank counterweight (18, 8).

(2) Although the radial engine with an even number of cylinders is completely balanced with respect to forces, with an odd number of cylinders the unbalanced force of lowest order is the $(p-1)$th.

(3) Apart from the unsteady component of zero order which is always present, the rolling moment is completely balanced with an even number of cylinders, and with an odd number is of order p or higher. With $R = 0$ or $k'^2 = s'(l - s')$ it is completely gone.

(4) If it is unnecessary to consider forces of sixth or higher order, all engines with six or more cylinders may be considered balanced.

These conclusions will require some modification later when engines with eccentrically attached connecting-rods are treated.

The tabulation opposite, which includes the two-cylinder engine for completeness, shows the orders of the unbalanced forces and rolling moments for various numbers of cylinders. (The forces of first order marked $\ominus$ and rolling moments marked $\bigcirc$ can be balanced.)

The unbalanced steady force of first order is a vector of magnitude $\frac{1}{2}p\omega^2(Q + 2Q^{(p)})$ which rotates with the crank; the unsteady force of first order is a vector of magnitude $\frac{1}{2}p\dot\omega(Q + 2Q^{(p)})$ which lags 90° behind the crank when $\dot\omega > 0$. The forces of higher order behave similarly but have angular velocities $\pm j\omega$.

Unfortunately, the radial engine with an even number of cylinders is suitable only for two-stroke use. The more important four-stroke engine

requires an odd number of cylinders in order to have constant firing intervals.

Unbalanced Forces ● ⊖ and Rolling Moments ○

No. cyl. \ Order	1	2	3	4	5	6	7	8	9	10	11	12
2	●											
3	⊖	●	○	●				●	○	●		
4	⊖											
5	⊖			●	○	●						
6	⊖											
7	⊖					●	○	●				
8	⊖											
9	⊖							●	○	●		
10	⊖											
11	⊖									●	○	●
12	⊖											

20. The Semi-regular Radial Engine. — Since the symmetric four-cylinder radial engine with suitable counterweights is perfectly balanced, engines with 8, 12, 16, . . . cylinders need not have constant angular cylinder spacings to be balanced. Instead, for complete balance it is sufficient if corresponding pairs of cylinders make angles of 90° and 180° with one another (Fig. 43) and have the necessary counterweights. In other words, such an engine can be thought of as being made up of several four-cylinder radial engines arbitrarily turned with respect to each other.

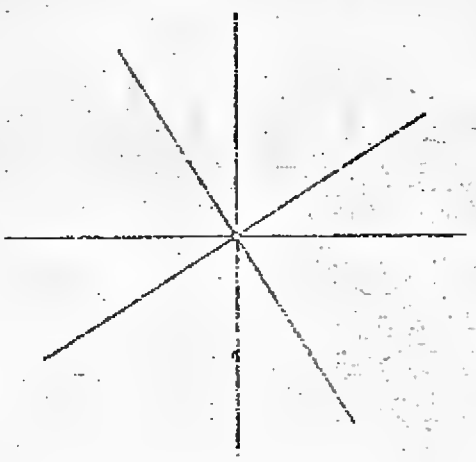


Fig. 43.—Semi-regular eight-cylinder radial engine.

On the other hand, if pairs of corresponding cylinders are at 180° but not at 90° to one another, all forces of higher order vanish (Sect. 17) but the first-order forces are only partly eliminated. This is true of the double V- or X-engine (Fig. 44). In order to find the inertia forces in this case we must add to (16. 5) the forces that are found by replacing ψ by $\psi + 180°$ and changing the signs of all terms in that equation. Then for the X-engine

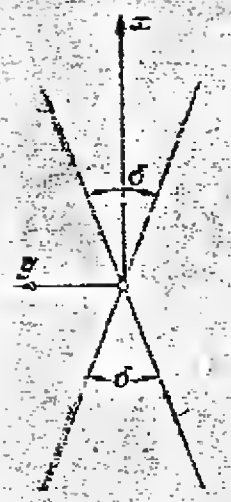


Fig. 44.—X- or double V-engine.

$$\left.\begin{aligned} X &= 2\omega^2[Q(1-\cos\delta) + 2Q^{(4)}]\cos\psi + ((\omega)). \\ Y &= 2\omega^2[Q(1-\cos\delta) + 2Q^{(4)}]\sin\psi + ((\omega)). \end{aligned}\right\} \quad (1)$$

Similarly, the rolling moment is

$$M_z = -4\omega P^{(4)}. \tag{2}$$

Here it is possible to eliminate the longitudinal force X by making $Q(1 - \cos\delta) + 2Q^{(4)} = 0$, or the transverse force Y by making $Q(1 - \cos\delta) + 2Q^{(4)} = 0$, or, finally, reduce both forces by making

$$Q[1 + (2\varkappa - 1)\cos\delta] + 2Q^{(4)} = 0, \tag{3}$$

where the factor $\varkappa$ lies between 0 and 1. We make it nearer 0 if the transverse force is to be the more nearly balanced out, or nearer to 1 when the longitudinal force is the more important, depending on the method of supporting the engine. Condition (3) becomes explicitly

$$sG + 2r\left\{\left(2 - \frac{s'}{l}[1 - (2\varkappa - 1)\cos\delta]\right)G' + [1 + (2\varkappa - 1)\cos\delta]G''\right\} = 0 \tag{4}$$

and is satisfied by means of a counterweight on the crank.

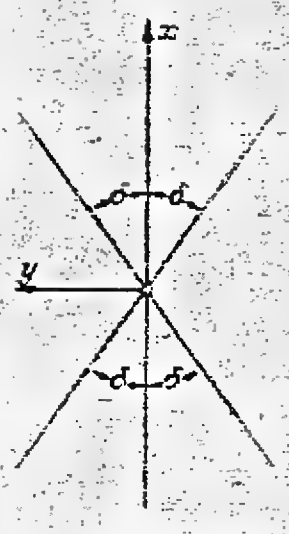


Fig. 45.—Double fan-type engine.

For the double fan-type engine (Fig. 45) we find from (18, 3)

$$\left.\begin{aligned} X &= 2\omega^2[Q(2 + \cos 2\delta) + 3Q^{(6)}]\cos\psi + (\!(\ddot{\omega})\!), \\ Y &= 2\omega^2[Q(1 - \cos 2\delta) + 3Q^{(6)}]\sin\psi + (\!(\dot{\omega})\!), \\ M_z &= -6\omega P^{(6)}. \end{aligned}\right\} \tag{5}$$

The balancing condition is

$$Q[(\varkappa + 1) + (2\varkappa - 1)\cos 2\delta] + 3Q^{(6)} = 0. \tag{6}$$

This leads to

$$sG - 2r\left\{\left(3 - \frac{s'}{l}[2 - \varkappa - (2\varkappa - 1)\cos 2\delta]\right)G' + [\varkappa + 1 + (2\varkappa - 1)\cos 2\delta]G''\right\} = 0, \tag{7}$$

which makes $Y = 0$ if $\varkappa = 0$ and $X = 0$ for $\varkappa = 1$.

§ 5. V-, Multi-row, and Radial Engines with Articulared Connecting-rods

21. Inertia Forces and Moments.—For an engine having a master connecting-rod and a number of somewhat dissimilar articulated rods, the analyses of V-, multi-row, and radial engines developed in § 4 can be considered as only first approximations, although satisfactory for many cases. Because of the difficulties involved in attaching a number of connecting-rods to a single crankpin, articulated rods are frequently used in V-engines (Fig. 46) and almost always in radial engines (Fig. 47). We shall now investigate the inertia forces and moments in this type of mechanism.[1]

[1] P. Ramstein, "Beitrag zur Theorie des Massenausgleiches von Sternmotoren", *Ing.-Arch.* I (1930), p. 16 and p. 245.

In order to avoid complicated formulæ we will confine our attention to forces and moments up to the second order.[1]

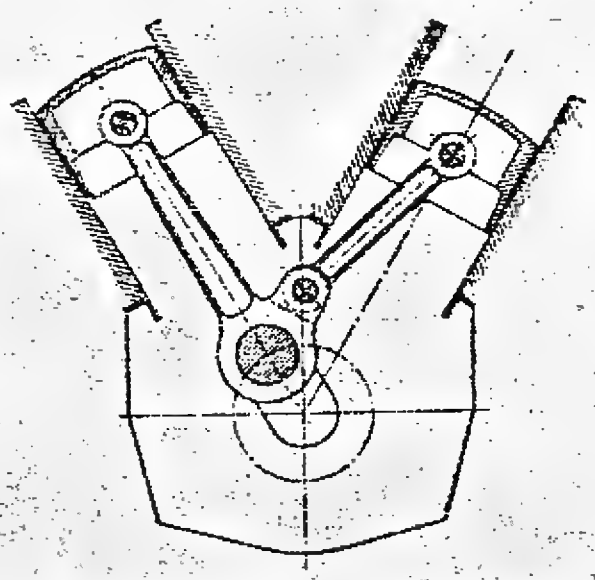


Fig. 46.—Crank mechanism of V-engine with articulated connecting-rod.

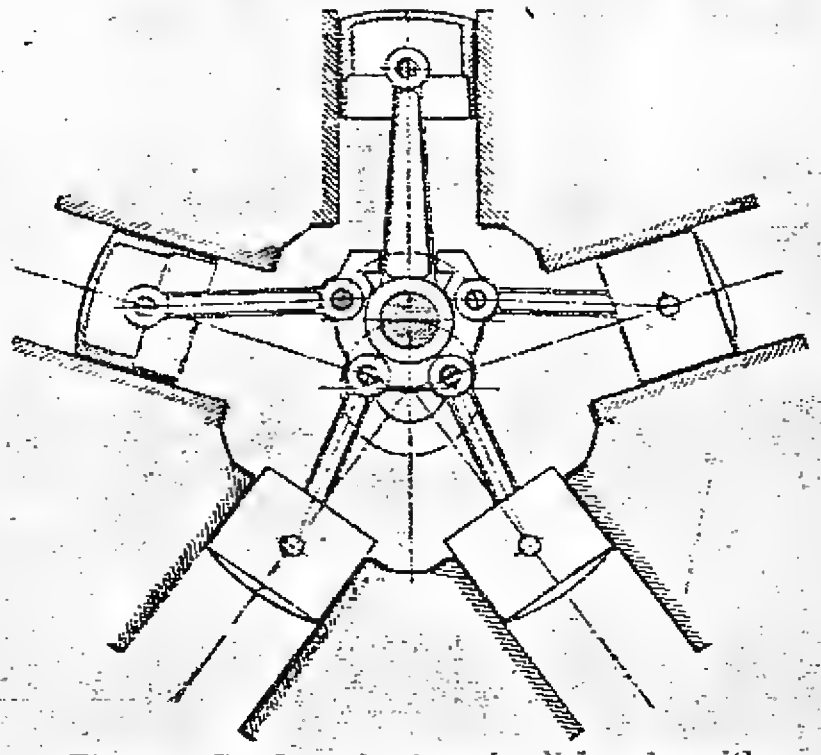


Fig. 47.—Crank mechanism of radial engine with articulated connecting-rods.

We keep the previous notation and designate the master connecting-rod or piston by the subscript 0, the ith articulated rod or piston by i, and the crank by using no subscript (Fig. 48). The angle between the master cylinder and the ith cylinder is δ_i. The articulated connecting-rod A_iB_i is attached to the master-rod A_0B_0 at the point A_i. The offset angle δ_i' of A_i with respect to the master-rod axis is generally made equal or nearly equal to δ_i. Since rather elaborate calculations have shown that making $\delta_i' = \delta_i$ is best for balancing, we will always take this to be the case. (Otherwise, all succeeding formulæ would be much more complicated and scarcely more accurate.) We also assume that the centres of gravity of the various connecting-rods S_0 and S_i lie on the axes of the corresponding rods A_0B_0 and A_iB_i. This is not always true, particularly for the master rod; investigations show,

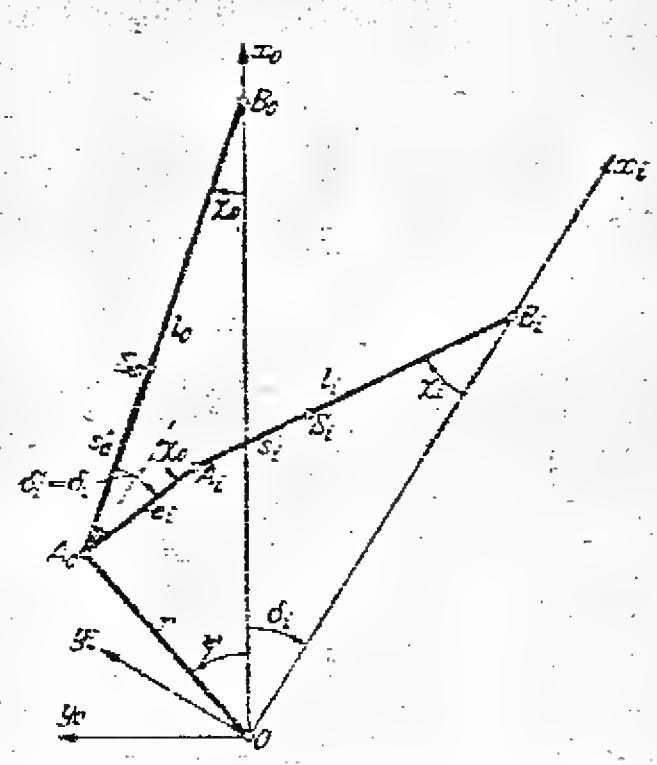


Fig. 48.—Master connecting-rod and ith articulated rod.

[1] With regard to forces and moments of higher order see A. Kramer, "Die freien Massenkräfte des Sternmotors", *Ing.-Arch.* 11 (1940), p. 424; "Zum Massenausgleich des Sternmotors", *Motorwelt Z.*, 4 (1942), p. 85; "Die erregenden Drehkräfte bei Flugmotoren mit mittelbarer Nebenpleuelanlenkung", *Luftfahrtforsch.* 18 (1941), p. 262 and p. 463.

however, that it would not be worth while to take account of the deviation.

For the master connecting-rod and the crank it follows from (5, 1) to (5, 3) or from (4, 6b) and (4, 10b) that, to the second order,

$$\left.\begin{aligned} X_s &= \omega^2[(Q_0 + Q_0')\cos\varphi + \lambda_0 Q_0 \cos 2\varphi], \\ Y_s &= \omega^2 Q_0' \sin\varphi, \\ M_{zs} &= \omega^2 R_0 \sin\varphi, \end{aligned}\right\} \tag{1}$$

where

$$\lambda_0 = \frac{r}{l_0},\quad Q_0 = \frac{r}{g}\left(\frac{\varepsilon_0'}{l_0}G_0' + G_0''\right),\quad Q_0' = \frac{1}{g}\left[\varepsilon G + r\left(1 - \frac{\varepsilon_0'}{l_0}\right)G_0'\right],\quad R_0 = \frac{\lambda_0}{g}[\varepsilon_0'(l_0 - \varepsilon_0') - k_0'^2]G_0'. \tag{2}$$

Here and in the future for simplicity we will omit the terms containing $\dot{\omega}$, because in any case they are much less important than the steady terms. (For the articulated-rod mechanisms they cannot be obtained as simply as before, i.e. directly from the ω^2 terms.)

For the ith articulated rod, x_i is the coordinate of the gudgeon-pin axis and x_i', y_i' the coordinates of the axis of the big-end bearing A_i. As in Sect. 3 we again divide the mass of the rod into two parts, placed at A_i and B_i. For the inertia forces in the (x_i, y_i)-system and the rolling moment of the articulated rod we obtain

$$\left.\begin{aligned} X_i &= -Q_i\frac{\bar{x}_i}{r} - Q_i'\frac{\bar{x}_i}{r}, \\ Y_i &= -Q_i'\frac{\bar{y}_i}{r}, \\ M_{zi} &= -R_i\frac{\bar{\chi}_i}{\lambda_i} - F_i^*\frac{1}{r^2}\frac{d}{di}(x_i'y_i' - y_i'x_i'), \end{aligned}\right\} \tag{3}$$

where

$$\left.\begin{aligned} \lambda_i &= \frac{r}{\bar{x}_i}, \qquad Q_i = \frac{r_i}{g}\left(\frac{\varepsilon_i'}{l_i}G_i' + G_i''\right), \qquad Q_i' = \frac{r}{g}\left(1 - \frac{\varepsilon_i'}{l_i}\right)G_i', \\ R_i &= \frac{\bar{x}_i}{g}[\varepsilon_i'(l_i - \varepsilon_i') - k_i'^2]G_i' \qquad F_i^* = rQ_i' = \frac{\lambda_i}{g}r(l_i - \varepsilon_i')G_i'. \end{aligned}\right\} \tag{4}$$

We have now to express the quantities x_i, x_i', y_i', χ_i and their derivatives in terms of the crank rotation φ. In order first to calculate x_i we observe that since $\delta_i' = \delta_i$ the slope of the line $A_0A_i = e_i$ (with respect to the x_i-axis) and is equal to the angle OB_0A_0 or to χ_0; we then use the following expressions which are obtained by inspection from Fig. 48:

$$\left.\begin{aligned} x_i &= r\cos(\varphi + \delta_i) + l_i\cos\chi_i + e_i\cos\chi_0, \\ r\sin(\varphi + \delta_i) &= l_i\sin\chi_i + e_i\sin\chi_0, \\ r\sin\varphi &= l_0\sin\chi_0. \end{aligned}\right\} \tag{5}$$

By eliminating χ_i and χ_0 and introducing the abbreviations

$$\bar{\lambda}_0 = \frac{r}{l_0}, \qquad \lambda_i = \frac{r}{l_i}, \qquad \varepsilon_i = \frac{e_i}{r} \tag{6}$$

we get the exact relation

$$\frac{x_i}{r} = \cos(\psi + \delta_i) + \frac{1}{\lambda_i}\{1 - [\lambda_i \sin(\psi + \delta_i) - \varepsilon_i \lambda_0 \lambda_i \sin\psi]^2\}^{1/2} + \varepsilon_i (1 - \lambda_0^2 \sin^2\psi)^{1/2}.$$

This expression can be developed approximately if we assume λ_0 and λ_i to be small quantities and also rewrite the products and squares of sine and cosine

$$\sin^2(\psi + \delta_i) = \frac{1}{2} - \frac{1}{2}\cos 2(\psi + \delta_i), \qquad \sin^2\psi = \frac{1}{2} - \frac{1}{2}\cos 2\psi,$$

$$\sin(\psi + \delta_i)\sin\psi = \frac{1}{2}\cos\delta_i - \frac{1}{2}\cos(2\psi + \delta_i);$$

we then find

$$\frac{x_i}{r} = a_i + \cos(\psi + \delta_i) + \frac{1}{4}\lambda_i \cos 2(\psi + \delta_i) - \frac{1}{2}\varepsilon_i \lambda_0 \lambda_i \cos(2\psi + \delta_i) + \frac{1}{4}\varepsilon_i \lambda_0^2 \cos 2\psi. \qquad (7)$$

The constant term a_i is unimportant. Differentiating this expression twice with respect to time and assuming ω constant, yields

$$-\frac{\ddot{x}_i}{r} = \omega^2[\cos\delta_i \cos\psi - \sin\delta_i \sin\psi + (\lambda_i \cos 2\delta_i - 2\varepsilon_i \lambda_0 \lambda_i \cos\delta_i + \varepsilon_i \lambda_0^2)\cos 2\psi - \\ - (\lambda_i \sin 2\delta_i - 2\varepsilon_i \lambda_0 \lambda_i \sin\delta_i)\sin 2\psi]. \qquad (8)$$

From Fig. 48 the coordinates of A_i are found

$$x_i' = r\cos(\psi + \delta_i) + \varepsilon_i \cos\chi_i, \qquad y_i' = r\sin(\psi + \delta_i) - \varepsilon_i \sin\chi_i.$$

If χ_i is eliminated by use of the third equation (5) we get

$$\left.\begin{aligned} \frac{x_i'}{r} &= \cos(\psi + \delta_i) + \varepsilon_i\left(1 - \frac{1}{4}\lambda_i^2 + \frac{1}{4}\lambda_i^2 \cos 2\psi\right), \\ \frac{y_i'}{r} &= \sin(\psi + \delta_i) - \varepsilon_i \lambda_0 \sin\psi. \end{aligned}\right\} \qquad (9)$$

From this follow

$$\left.\begin{aligned} -\frac{\ddot{x}_i'}{r} &= \omega^2(\cos\delta_i \cos\psi - \sin\delta_i \sin\psi + \varepsilon_i \lambda_i^2 \cos 2\psi), \\ -\frac{\ddot{y}_i'}{r} &= \omega^2[\sin\delta_i \cos\psi + (\cos\delta_i - \varepsilon_i \lambda_0)\sin\psi]. \end{aligned}\right\} \qquad (10)$$

By combining (9) and (10) and limiting ourselves to terms of zero and first order (which suffices here) we get

$$-\frac{1}{r^2}\frac{d}{d\psi}(x_i' y_i' - y_i' x_i') = -\frac{1}{r^2}(x_i' \ddot{y}_i' - y_i' \ddot{x}_i') = \omega^2 \varepsilon_i[\sin\delta_i \cos\psi + (\cos\delta_i - \varepsilon_i \lambda_0)\sin\psi]. \qquad (11)$$

Finally, in order to calculate $\ddot{\chi}_i$ we combine the second and third of equations (5) to obtain

$$\sin\chi_i = \lambda_i \sin(\psi + \delta_i) - \varepsilon_i \lambda_0 \lambda_i \sin\psi. \qquad (12)$$

Differentiation yields

$$\dot{\chi}_i = \omega\lambda_i[\cos(\psi + \delta_i) - \varepsilon_i\lambda_0\cos\psi]\frac{1}{\cos\chi_i}.$$

The last factor $1/\cos\chi_i$ may be dropped, in view of the order of accuracy sought, and there results

$$\frac{\ddot{\chi}_i}{\lambda_i} = -\omega^2[\sin\delta_i\cos\psi + (\cos\delta_i - \varepsilon_i\lambda_0)\sin\psi]. \tag{13}$$

Next, the forces X_i, Y_i and the rolling moment M_{xi} (3) can be written out in full:

$$\left.\begin{aligned} X_i &= \omega^2\{(Q_i + Q_i')[\cos\delta_i\cos\psi - \sin\delta_i\sin\psi + \varepsilon_i\lambda_0^2\cos 2\psi) + \\ &\quad + \lambda_i Q_i[(\cos 2\delta_i - 2\varepsilon_i\lambda_0\cos\delta_i)\cos 2\psi - (\sin 2\delta_i - 2\varepsilon_i\lambda_0\sin\delta_i)\sin 2\psi]\}, \\ Y_i &= \omega^2 Q_i'[\sin\delta_i\cos\psi + (\cos\delta_i - \varepsilon_i\lambda_0)\sin\psi], \\ M_{xi} &= \omega^2(R_i + \varepsilon_i R_i^*)[\sin\delta_i\cos\psi + (\cos\delta_i - \varepsilon_i\lambda_0)\sin\psi]. \end{aligned}\right\} \tag{14}$$

Expressions (1) and (14) together give the inertia forces and moments in V-, multi-row, and radial engines.

22. The V-engine.—For the V-engine we use the symbols of Fig. 48, replacing the subscript i by 1, getting

$$\left.\begin{aligned} X &= X_0 + X_1\cos\delta_1 + Y_1\sin\delta_1, \\ Y &= Y_0 - X_1\sin\delta_1 + Y_1\cos\delta_1, \\ M_x &= M_{x0} - M_{x1}. \end{aligned}\right\} \tag{1}$$

The quantities (21, 1) and (21, 14) must still be inserted here. Instead of using the general expressions that were obtained, we limit ourselves to the cases $\delta_1 = 90°$ and $\delta_1 = 180°$, the only ones that offer definite possibilities for balancing.

For the V-angle $\delta_1 = 90°$ there results

$$\left.\begin{aligned} X &= \omega^2[(Q_0 + Q_0' + Q_1')\cos\psi - \varepsilon_1\lambda_0 Q_1'\sin\psi + \lambda_0 Q_0\cos 2\psi], \\ Y &= \omega^2[(Q_1 + Q_0' + Q_1')\sin\psi + (\lambda_1 Q_1 - \varepsilon_0\lambda_0^2 Q_0 - \varepsilon_1\lambda_0^2 Q_1')\cos 2\psi - 2\varepsilon_1\lambda_0\lambda_1 Q_1\sin 2\psi], \\ M_x &= \omega^2[(R_1 + \varepsilon_1 R_1^*)\cos\psi + (R_0 - \varepsilon_1\lambda_0 R_1 - \varepsilon_1^2\lambda_0 R_1^*)\sin\psi]. \end{aligned}\right\} \tag{2}$$

It can be seen that since $Q_1' \neq 0$, complete first-order balancing is not possible. However, if we do not insist on compensating the rather small term $-\varepsilon_1\lambda_0 Q_1'\sin\psi$ it is only necessary to make $Q_0 = Q_1$ and then to eliminate the other first-order terms by using a counterweight. The counterweight must be selected to satisfy the relation

$$Q_0 + Q_0' + Q_1' = 0 \tag{3}$$

or, according to (21, 2) and (21, 4), the condition

$$sG + r\left[G_0' + \left(1 - \frac{s_1'}{l_1}\right)G_1' + G_0''\right] = 0. \tag{4}$$

The other condition, $Q_0 = Q_1$, means that the reduced weights (with respect to the gudgeon pins) are equal:

$$\frac{s_0}{l_0}G_0' + G_0'' = \frac{s_1'}{l_1}G_1' + G_1''. \tag{5}$$

Condition (5) is easily satisfied by a suitable choice of piston weights, while (4) requires a very-heavy counterweight.

The rolling moment of first-order vanishes provided, first, that the master connecting-rod is so shaped that

$$R_0 = 0, \quad \text{therefore} \quad \frac{s_0^{*2}}{s_0'} = s_0'(l_0 - s_0'), \tag{6}$$

and, second, that the articulated rod satisfies

$$R_1 + \varepsilon_1 R_1^* = 0, \quad \text{therefore} \quad \frac{s_1^{*2}}{s_1' + \varepsilon_1} = (s_1' + \varepsilon_1)(l_1 - s_1'). \tag{7}$$

Both conditions are easily met (Sect. 6).

For the "opposed" V-engine (or two-cylinder radial engine) $\delta_1 = 180°$ and we get

$$\left.\begin{aligned} X &= \omega^2[(Q_0 + Q_1 + Q_0' + Q_1')\cos\psi + \\ &\quad + (\lambda_0 Q_0 - \lambda_1 Q_1 - 2\varepsilon_1\lambda_0\lambda_1 Q_1 - \varepsilon_1\lambda_0^2 Q_1 - \varepsilon_1\lambda_0^2 Q_1')\cos 2\psi]. \\ Y &= \omega^2(Q_0' + Q_1' + \varepsilon_1\lambda_0 Q_1')\sin\psi. \\ M_z &= \omega^2[R_0 - (1 + \varepsilon_1\lambda_0)(R_1 + \varepsilon_1 R_1^*)]\sin\psi. \end{aligned}\right\} \tag{8}$$

The force of first order is not completely balanced. However, by means of a counterweight that satisfies the condition

$$\varkappa(Q_0 + Q_1 - \varepsilon_1\lambda_0 Q_1') + (Q_0' + Q_1' + \varepsilon_1\lambda_0 Q_1') = 0 \tag{9}$$

or

$$\begin{aligned} sG + r\Big\{&\left[1 + (\varkappa - 1)\frac{s_0'}{l_0}\right]G_0' + \\ &+ \left[1 + (1 - \varkappa)\varepsilon_1\lambda_0 + (\varkappa - 1)(1 + \varepsilon_1\lambda_0)\frac{s_1'}{l_1}\right]G_1' + \varkappa(G_0'' + G_1'')\Big\} = 0 \end{aligned} \tag{10}$$

we can eliminate the lateral force Y by making $\varkappa = 0$, eliminate the first-order longitudinal force X by making $\varkappa = 1$, or reduce both with an intermediate value of $\varkappa$. The second-order longitudinal force vanishes when

$$\lambda_0 Q_0 - \lambda_1 Q_1 - 2\varepsilon_1\lambda_0\lambda_1 Q_1 - \varepsilon_1\lambda_0^2 Q_1 - \varepsilon_1\lambda_0^2 Q_1' = 0. \tag{11}$$

These relations among Q_0, Q_1, Q_1', λ_0, λ_1 and ε_1 can be satisfied without encountering structural difficulties. For example, if $Q_0 = Q_1 = 3Q_1'$ is

specified, then for $\lambda_0 = \frac{1}{4}$ and $\varepsilon_1 = \frac{1}{3}$ there results $\lambda_1 = 1/5{\cdot}25$. This leads to $l_0 = 4r$, $e_1 = r/3$, $l_1 + e_1 = 5{\cdot}6r$, a usable solution.

Finally, the rolling moment vanishes if the conditions (6) and (7) are satisfied; these, by the way, may be replaced here by the single more general condition:

$$R_0 = (1 + \varepsilon_1 \lambda_0)(P_1 + \varepsilon_1 P_1^*). \tag{12}$$

We can easily show that if $\varepsilon_1 = 0$ the conditions (4) to (7) and (9) to (12) reduce to the results given in Sect. 16 and 17.

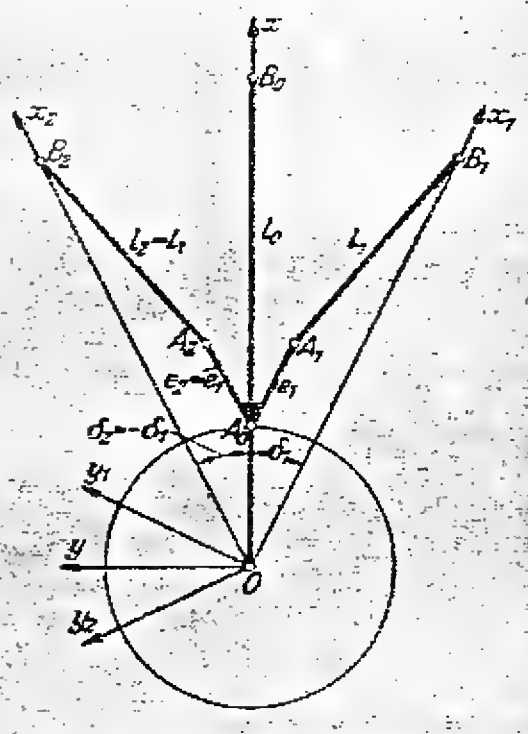


Fig. 49.—Three-row engine (fan-type).

23. The Fan-type Engine.—For the three-cylinder fan-type engine we use the symbols given in Fig. 49. We assume the mechanisms of the articulated connecting-rods to be identical and therefore write $Q_2 = Q_1$, $Q_2' = Q_1'$, $R_2 = R_1$, $R_2^* = R_1^*$, $\varepsilon_2 = \varepsilon_1$, $\delta_2 = -\delta_1$ and $\lambda_2 = \lambda_1$; we then get

$$\left.\begin{aligned} X &= X_0 + (X_1 + X_2)\cos\delta_1 + (Y_1 - Y_2)\sin\delta_1, \\ Y &= Y_0 - (X_1 - X_2)\sin\delta_1 + (Y_1 + Y_2)\cos\delta_1, \\ M_x &= M_{x0} + M_{x1} + M_{x2}. \end{aligned}\right\} \tag{1}$$

When we introduce (21, 1) and (21, 14) into the above equations we get the useful relations:

$$\left.\begin{aligned} X &= \omega^2\{(Q_0 + Q_0' + 2Q_1\cos^2\delta_1 + 2Q_1')\cos\psi + [\lambda_0 Q_0 + 2Q_1\cos\delta_1(\lambda_1\cos 2\delta_1 + \\ &\qquad + \varepsilon_1\lambda_0^2 - 2\varepsilon_1\lambda_0\lambda_1\cos\delta_1) + 2\varepsilon_1\lambda_0^2 Q_1'\cos\delta_1]\cos 2\psi\}, \\ Y &= \omega^2\{[Q_0' + 2Q_1\sin^2\delta_1 + 2Q_1'(1 - \varepsilon_1\lambda_0\cos\delta_1)]\sin\psi + 2\lambda_1 Q_1\sin\delta_1(\sin 2\delta_1 - \\ &\qquad - 2\varepsilon_1\lambda_0\sin\delta_1)\sin 2\psi\}, \\ M_x &= \omega^2[R_0 + 2(R_1 + \varepsilon_1 R_1^*)(\cos\delta_1 - \varepsilon_1\lambda_0)]\sin\psi. \end{aligned}\right\} \tag{2}$$

As in the case discussed in Sect. 18, balancing is possible only for $\delta_1 = 60°$ (and also for $\delta_1 = 120°$, which is to be taken up in Sect. 24). For $\delta_1 = 60°$ equation (2) becomes

$$\left.\begin{aligned} X &= \omega^2[(Q_0 + Q_0' + \tfrac{1}{2}Q_1 + 2Q_1')\cos\psi + (\lambda_1 Q_0 - \tfrac{1}{2}\lambda_1 Q_1 + \varepsilon_1\lambda_0^2 Q_1 - \\ &\qquad - \varepsilon_1\lambda_0\lambda_1 Q_1 + \varepsilon_1\lambda_0^2 Q_1')\cos 2\psi], \\ Y &= \omega^2[(Q_0' + \tfrac{3}{2}Q_1 - 2Q_1' - \varepsilon_1\lambda_0 Q_1')\sin\psi + \tfrac{3}{2}\lambda_1(1 - 2\varepsilon_1\lambda_0)Q_1\sin 2\psi], \\ M_x &= \omega^2[R_0 + (1 - 2\varepsilon_1\lambda_0)(P_1 + \varepsilon_1 P_1^*)]\sin\psi. \end{aligned}\right\} \tag{3}$$

For the first-order forces to vanish it is necessary that

$$Q_0 + Q_0' + \tfrac{1}{2}Q_1 + 2Q_1' = 0 \tag{4}$$

and that

$$Q_0' + \tfrac{3}{2}Q_1 + 2Q_1' - \varepsilon_1\lambda_0 Q_1' = 0.$$

From these, by subtraction, there follows

$$Q_1 = Q_0 + \varepsilon_1 \lambda_0 Q_1' \tag{5}$$

Conditions (4) and (5) lead to

$$sG + r\left[G_0' + G_0'' + \left(2 - \frac{3}{2}\frac{s_1'}{l_1}\right)G_1' + \frac{1}{2}G_1''\right] = 0, \tag{6}$$

$$\left[\frac{s_1'}{l_1} - \varepsilon_1\lambda_0\left(1 - \frac{s_1'}{l_1}\right)\right]G_1' + G_1'' = \frac{s_0'}{l_0}G_0' + G_0''. \tag{7}$$

The first of these demands a counterweight on the crank, while the second requires that the reduced weights of the mechanisms of the articulated connecting-rods (with respect to the gudgeon-pins) be somewhat larger than that of the master-rod mechanism.

If conditions (4) and (5) are satisfied and the quantities Q_0 and Q_1' eliminated by means of (5), the remaining second-order forces X and Y are

$$\left.\begin{aligned} X &= \omega^2(\lambda_0 - \tfrac{1}{2}\lambda_1 + \varepsilon_1\lambda_0^2 - \varepsilon_1\lambda_0\lambda_1)Q_1\cos 2\psi, \\ Y &= \tfrac{3}{2}\omega^2\lambda_1(1 - 2\varepsilon_1\lambda_0)Q_1\sin 2\psi. \end{aligned}\right\} \tag{8}$$

In order to eliminate these second-order forces we must satisfy simultaneously the conditions

$$\lambda_0 - \tfrac{1}{2}\lambda_1 + \varepsilon_1\lambda_0^2 - \varepsilon_1\lambda_0\lambda_1 = 0 \quad \text{and} \quad 1 - 2\varepsilon_1\lambda_0 = 0. \tag{9}$$

From these

$$\lambda_1 = \frac{3}{2}\lambda_0 \quad \text{and} \quad \varepsilon_1 = \frac{1}{2\lambda_0}. \tag{10}$$

In order to get an idea of the usefulness of these conditions we introduce the quantity

$$\lambda_1' = \frac{r}{l_1 + e_1} = \frac{\lambda_1}{1 + \varepsilon_1\lambda_1} \tag{11}$$

whose value is, from (10),

$$\lambda_1' = \tfrac{6}{5}\lambda_0. \tag{12}$$

Such values are quite possible; the values of ε_1 from (10), on the other hand, are rather large. These will be used only when balancing both second-order forces offers great advantages. If we are satisfied with eliminating only the longitudinal force we need only use the simpler first condition (9), from which there results

$$\lambda_1 = \lambda_0\frac{1 + \varepsilon_1\lambda_0}{\frac{1}{2} + \varepsilon_1\lambda_0} \quad \text{and} \quad \lambda_1' = \lambda_0\frac{1 + \varepsilon_1\lambda_0}{\frac{1}{2} + \varepsilon_1\lambda_0(2 + \varepsilon_1\lambda_0)}. \tag{13}$$

For example, if $\lambda_0 = 1/3$ and $\varepsilon_1 = 1$ are specified then $\lambda_1 = 1/2{\cdot}9$; therefore $l_0 = 3r$, $e_1 = r$, $l_1 + e_1 = 2{\cdot}9r$, which is a satisfactory solution.

The first-order rolling moment M_z disappears when we satisfy either

the conditions (22, 6) and (22, 7) or the more general relation

$$R_0 + (1 - 2\varepsilon_1 \lambda_0)(R_1 + \varepsilon_1 R_1^*) = 0. \tag{14}$$

Again, we can easily show that if $\varepsilon_1 = 0$, all the results reduce to those of Sect. 18.

24. The Radial Engine.—We assume first that all $p - 1$ articulated mechanisms are alike so that in the quantities Q_i, Q_i', R_i, R_i^*, λ_i, ε_i the subscripts 2, 3, 4 . . . $p - 1$ may be replaced by 1. Also the angle included between each pair of consecutive cylinders is $\delta = 2\pi/p$ (Fig. 50), so that

$$\delta_i = i\delta = i\frac{2\pi}{p} \qquad (i = 1, 2, \ldots, p-1). \tag{1}$$

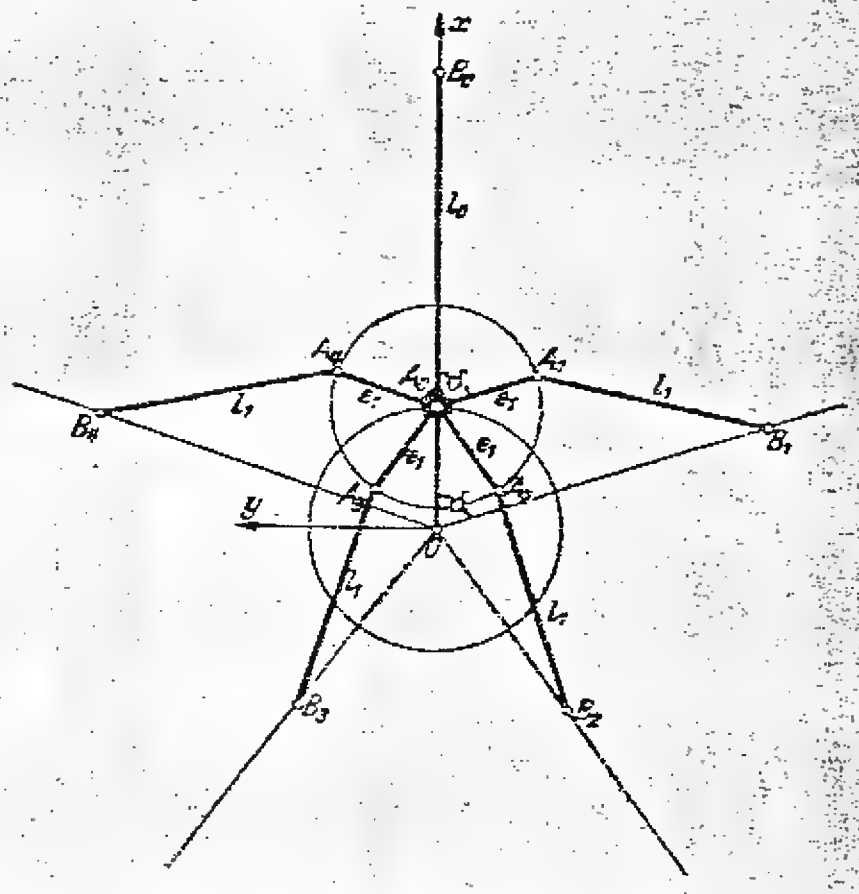


Fig. 50.—Five-cylinder radial engine.

Then

$$\left.\begin{aligned} X &= X_0 + \sum_{i=1}^{p-1} X_i \cos\delta_i - \sum_{i=1}^{p-1} Y_i \sin\delta_i, \\ Y &= Y_0 - \sum_{i=1}^{p-1} X_i \sin\delta_i + \sum_{i=1}^{p-1} Y_i \cos\delta_i, \\ M_z &= M_{z0} + \sum_{i=1}^{p-1} M_{zi}. \end{aligned}\right\} \tag{2}$$

If the expressions (21, 1) and (21, 14) are introduced here, the following summations will result, which can be evaluated beforehand by use of (10, 4a)

and (10, 4b) provided the summation index i is first replaced by $k-1$,

$$\sum_{i=1}^{p-1}\cos\delta_i=\sum_{i=1}^{p-1}\cos i\frac{2\pi}{p}=\sum_{k=2}^{p}\cos(k-1)\frac{2\pi}{p}=\sum_{k=1}^{p}\cos(k-1)\frac{2\pi}{p}-1=-1,$$

$$\sum_{i=1}^{p-1}\sin\delta_i=\sum_{k=2}^{p}\sin(k-1)\frac{2\pi}{p}=\sum_{k=1}^{p}\sin(k-1)\frac{2\pi}{p}=0,$$

$$\sum_{i=1}^{p-1}\cos^2\delta_i=\frac{1}{2}(p-1)+\frac{1}{2}\sum_{i=1}^{p-1}\cos 2\delta_i=\frac{p}{2}-1\quad\text{for } p>2,$$

$$\sum_{i=1}^{p-1}\sin^2\delta_i=\frac{p}{2}\quad\text{for } p>2,$$

$$\sum_{i=1}^{p-1}\sin\delta_i\cos\delta_i=0\quad\text{for } p>2,$$

$$\sum_{i=1}^{p-1}\cos\delta_i\cos 2\delta_i=\frac{1}{2}\sum_{i=1}^{p-1}\cos 3\delta_i+\frac{1}{2}\sum_{i=1}^{p-1}\cos\delta_i=\begin{cases}+\frac{1}{2}\\-1\end{cases}\quad\text{for}\quad p\begin{cases}=3\\>3,\end{cases}$$

$$\sum_{i=1}^{p-1}\cos\delta_i\sin 2\delta_i=0,$$

$$\sum_{i=1}^{p-1}\sin\delta_i\cos 2\delta_i=0,$$

$$\sum_{i=1}^{p-1}\sin\delta_i\sin 2\delta_i=\begin{cases}-\frac{3}{2}\\0\end{cases}\quad\text{for } p\begin{cases}=3.\\>3.\end{cases}$$

Putting these values into (2), we get for $p>2$,

$$\left.\begin{aligned}X&=\omega^2\left\{\left[Q_0+Q_0'+\left(\frac{p}{2}-1\right)Q_1+(p-1)Q_1'\right]\cos\varphi+\right.\\&\quad\left.+\left[\lambda_0 Q_0+\begin{cases}+\frac{\lambda}{2}\\-1\end{cases}\lambda_1 Q_1-\varepsilon_2\lambda_0^2(Q_1+Q_1')-(p-2)\varepsilon_1\lambda_0\lambda_1 Q_1\right]\cos 2\varphi\right\},\\Y&=\omega^2\left\{\left[Q_0'+\frac{p}{2}Q_1+(p-1+\varepsilon_1\lambda_0)Q_1'\right]\sin\varphi-\right.\\&\qquad\left.-\lambda_1\left[\begin{cases}\frac{3}{2}\\0\end{cases}+p\varepsilon_2\lambda_0\right]Q_1\sin 2\varphi\right\},\\Y_2&=\omega^2\{P_0-[1+(p-1)\varepsilon_1\lambda_0](P_1+\varepsilon_1 P_1^*)\}\sin\varphi.\end{aligned}\right\}\quad\text{for } p\begin{cases}=3\\>3\end{cases}\qquad(3)$$

In order for the first-order forces to vanish the two following equations must be satisfied:

$$Q_0+Q_0'+\left(\frac{p}{2}-1\right)Q_1+(p-1)Q_1'=0,\qquad(4)$$

$$Q_0'+\frac{p}{2}Q_1+(p-1+\varepsilon_1\lambda_0)Q_1'=0.$$

By subtraction we get

$$Q_1=Q_0-\varepsilon_1\lambda_0 Q_1'.\qquad(5)$$

Conditions (4) and (5) yield explicitly

$$\varepsilon G + r\left[G_2' + G_0'' + \left(p - 1 - \frac{p}{2}\frac{\varepsilon_2}{\lambda_2}\right)G_1' + \left(\frac{p}{2} - 1\right)G_1''\right] = 0, \tag{6}$$

$$\left[\frac{\varepsilon_2'}{\lambda_1} + \varepsilon_1\lambda_0\left(1 - \frac{\varepsilon_1'}{\lambda_1}\right)\right]G_1' + G_1'' = \frac{\varepsilon_0'}{\lambda_0}G_0' + G_0''. \tag{7}$$

The first requires a counterweight on the crank; the second demands that the reduced weights of the articulated-rod mechanisms at the gudgeon-pins be somewhat less than that of the master rod.

If conditions (4) and (5) are satisfied, by eliminating Q_0 and Q_1' by means of (5), it is possible to represent the remaining second-order forces X and Y as follows:

$$\left.\begin{aligned} X &= \omega^2(\lambda_0 + \tfrac{1}{2}\lambda_1 - \varepsilon_2\lambda_0^2 - \varepsilon_1\lambda_0\lambda_1)Q_1\cos 2\psi, \\ Y &= -\tfrac{3}{2}\omega^2(1 + 2\varepsilon_1\lambda_0)\lambda_1 Q_1\sin 2\psi, \end{aligned}\right\}\ \text{for } p = 3;$$

$$\left.\begin{aligned} X &= \omega^2(\lambda_0 - \lambda_2 - \varepsilon_1\lambda_0^2 - (p-2)\varepsilon_1\lambda_0\lambda_2)Q_1\cos 2\psi, \\ Y &= -p\omega^2\varepsilon_1\lambda_0\lambda_1 Q_1\sin 2\psi, \end{aligned}\right\}\ \text{for } p > 3. \tag{8}$$

In order to obtain second-order balancing in the case $p = 3$ the following equations must be satisfied:

$$\lambda_0 + \tfrac{1}{2}\lambda_1 - \varepsilon_1\lambda_0^2 - \varepsilon_1\lambda_0\lambda_1 = 0 \quad \text{and} \quad 1 + 2\varepsilon_1\lambda_0 = 0.$$

The first of these leads to negative values of λ_0 or λ_1, the second to negative values of ε_1 or λ_0. This shows that balancing the second-order forces in the three-cylinder radial engine is impossible (see Sect. 19).

On the other hand, if $p > 3$, X, at least, can be eliminated by satisfying the relation

$$\lambda_0 - \lambda_1 - \varepsilon_1\lambda_0^2 - (p-2)\varepsilon_1\lambda_0\lambda_2 = 0. \tag{9}$$

From this follow

$$\lambda_1 = \lambda_0\frac{1 - \varepsilon_1\lambda_0}{1 + (p-2)\varepsilon_1\lambda_0} \quad \text{and} \quad \lambda_1' = \lambda_0\frac{1 - \varepsilon_1\lambda_0}{1 + \varepsilon_1\lambda_0(p - 1 - \varepsilon_1\lambda_0)}. \tag{10}$$

For example, if $p = 5$, $\lambda_0 = \frac{1}{4}$ and $\varepsilon_1 = \frac{1}{8}$, then $\lambda_1 = 1/5{\cdot}4$ and $\lambda_1' = 1/5{\cdot}8$. It is apparent that λ_0 and λ_1' must differ considerably, a condition that may be difficult to obtain in construction. With an increasing number of cylinders, equations (10) lead to still more difficult conditions, so that the prospects for balancing the second-order forces X, even for $p > 3$, are not encouraging. The second-order force Y cannot be eliminated in any case; however, because of the factor $\varepsilon_1\lambda_0\lambda_1$ it is comparatively small.

Again, the first-order rolling moment M_z vanishes if R_0 and $R_1 + \varepsilon_1 R_1^*$ are zero (as in (22, 6) and 22, 7)) or if the condition

$$R_0 = [1 + (p-1)\varepsilon_2\lambda_0](R_1 + \varepsilon_1 R_1^*) \tag{11}$$

is satisfied.

On the whole, it must be concluded that with articulated connecting-rods inertia balancing of the radial engine, especially with many cylinders, becomes much more difficult. However, the use of several banks of cylinders offers real possibilities for balancing, and will be discussed next.

§ 6. V-, Fan-like in Banks and Radial Engines

25. Multi-bank Engines with several Cylinders per Bank.—The inertia forces and moments in V- and fan-type engines can be expressed in the general form:

$$\left.\begin{aligned} X &= \omega^2(q_1 \cos\psi + q_2 \cos 2\psi - q_4 \cos 4\psi + - \cdots) + ((\omega)), \\ Y &= \omega^2(r_1 \sin\psi + r_2 \sin 2\psi - r_4 \sin 4\psi + - \cdots) + ((\omega)), \\ M_z &= \omega^2(s_1 \sin\psi - s_3 \sin 3\psi + - \cdots) + ((\omega)) - p\omega R^{(p)}, \end{aligned}\right\} \tag{1}$$

in which the coefficients q_i, r_i, s_i are found from the quantities Q, $Q^{(p)}$, R, A_{2i}, C_{2i-1} and δ as already described. The x-axis is the axis of symmetry (usually vertical). According to (23, 2), these formulas hold for fan-type engines with articulated connecting-rods provided $\dot{\omega} = 0$, but do not hold for V-engines of this type. The cylinder-number p of a V- or fan-type engine is 2, 3, 4 . . .

Since equations (1) are exactly the same as for the one-cylinder engine (5, 1) to (5, 3), with the only difference that for the latter the Y-coefficients r_2, r_4 . . . are zero, we can extend all the results of Sect. 7 to 13 (generalized to agree with the more general form of the expression for Y) to V- and fan-type engines with more than one cylinder per row. Thus, for the two-stroke V-engine, or fan-type engine with two cylinders per row, and with 180° crank-angle, if we let $\dot{\omega} = 0$ for simplicity and designate by a the spacing of cylinders in each row, we get

$$\left.\begin{aligned} X &= 2\omega^2(q_2 \cos 2\psi - q_4 \cos 4\psi + - \cdots), \\ Y &= 2\omega^2(r_2 \sin 2\psi - r_4 \sin 4\psi + - \cdots). \\ M_x &= -\omega^2 a r_1 \sin\psi, \\ M_y &= \omega^2 a q_1 \cos\psi, \\ M_z &= 0. \end{aligned}\right\} \tag{2}$$

In the same way, for two- or four-stroke V- and fan-type engines having 120° crank-angles and with three cylinders per row (the cranks being numbered according to Fig. 8 in Sect. 8) we get

$$\left.\begin{aligned} X &= 3\omega^2(q_6 \cos 6\psi - q_{12} \cos 12\psi + - \cdots), \\ Y &= 3\omega^2(r_6 \sin 6\psi - r_{12} \sin 12\psi + - \cdots), \\ M_x &= -\sqrt{3}\omega^2 a(r_1 \cos\psi - r_2 \cos 2\psi - r_4 \cos 4\psi + r_8 \cos 8\psi + r_{10} \cos 10\psi - - + + \cdots), \\ M_y &= -\sqrt{3}\omega^2 a(q_1 \sin\psi - q_2 \sin 2\psi - q_4 \sin 4\psi + q_8 \sin 8\psi + q_{10} \sin 10\psi - - + + \cdots), \\ M_z &= -3\omega^2(s_2 \sin 2\psi - s_8 \sin 9\psi + - \cdots). \end{aligned}\right\} \tag{3}$$

In general, for the V- or fan-type engine with n cylinders per row we find, analogous to (10, 5), (10, 7), (10, 8), (10, 10), (10, 8a), (10, 10a), and (11, 2), (11, 3):

(a) Two-stroke (crank-angle $= 2\pi/n$):

$$\left.\begin{aligned}
&X = n\omega^2(\pm q_n \cos n\psi - q_{2n} \cos 2n\psi \pm q_{3n} \cos 3n\psi - \cdots)\\
&Y = n\omega^2(\pm r_n \sin n\psi - r_{2n} \sin 2n\psi \pm r_{3n} \sin 3n\psi - \cdots)
\end{aligned}\right\} \text{ for } n = \begin{cases} 2, 6, 10, \ldots \\ 4, 8, 12, \ldots, \end{cases}$$

$$\left.\begin{aligned}
&X = n\omega^2(q_{2n} \cos 2n\psi - q_{4n} \cos 4n\psi + - \cdots)\\
&Y = n\omega^2(r_{2n} \sin 2n\psi - r_{4n} \sin 4n\psi + - \cdots)
\end{aligned}\right\} \text{ for } n = 3, 5, 7, \ldots,$$

$$\left.\begin{aligned}
&M_x = -\omega^2 a[\alpha_1 r_1 \sin(\psi + \beta_1) + \alpha_2 r_2 \sin(2\psi + \beta_2) + \cdots]\\
&M_y = \omega^2 a[\alpha_1 q_1 \cos(\psi + \beta_1) + \alpha_2 q_2 \cos(2\psi + \beta_2) + \cdots]
\end{aligned}\right\} \text{ for } n = 2, 3, 4, \ldots,$$

$$M_z = 0 \quad \text{for } n = 2, 4, 6, \ldots,$$

$$M_z = \mp n\omega^2(s_n \sin n\psi - s_{3n} \sin 3n\psi + - \cdots) \quad \text{for } n = \begin{cases} 3, 7, 11, \ldots \\ 5, 9, 13, \ldots, \end{cases}$$

(4)

(b) Four-stroke (crank-angle $= 4\pi/n$):

$$\left.\begin{aligned}
&X = n\omega^2\left(\pm q_{\frac{n}{2}} \cos\frac{n}{2}\psi - q_n \cos n\psi \pm q_{\frac{3n}{2}} \cos\frac{3n}{2}\psi - \cdots\right)\\
&Y = n\omega^2\left(\pm r_{\frac{n}{2}} \sin\frac{n}{2}\psi - r_n \sin n\psi \pm r_{\frac{3n}{2}} \sin\frac{3n}{2}\psi - \cdots\right)
\end{aligned}\right\} \text{ for } n = \begin{cases} 4, 12, 20, \ldots \\ 8, 16, 24, \ldots, \end{cases}$$

$$\left.\begin{aligned}
&X = n\omega^2(q_n \cos n\psi - q_{2n} \cos 2n\psi + - \cdots)\\
&Y = n\omega^2(r_n \sin n\psi - r_{2n} \sin 2n\psi + - \cdots)
\end{aligned}\right\} \text{ for } n = 6, 10, 14, \ldots,$$

$$\left.\begin{aligned}
&X = n\omega^2(q_{2n} \cos 2n\psi - q_{4n} \cos 4n\psi + - \cdots)\\
&Y = n\omega^2(r_{2n} \sin 2n\psi - r_{4n} \sin 4n\psi + - \cdots)
\end{aligned}\right\} \text{ for } n = 3, 5, 7, \ldots,$$

$$\left.\begin{aligned}
&M_x = 0\\
&M_y = 0
\end{aligned}\right\} \text{ for } n = 2, 4, 6, \ldots,$$

$$\left.\begin{aligned}
&M_x = -\omega^2 a[\alpha_1 r_1 \sin(\psi + \beta_1) + \alpha_2 r_2 \sin(2\psi + \beta_2) + \cdots]\\
&M_y = \omega^2 a[\alpha_1 q_1 \cos(\psi + \beta_1) + \alpha_2 q_2 \cos(2\psi + \beta_2) + \cdots]
\end{aligned}\right\} \text{ for } n = 3, 5, 7, \ldots,$$

$$M_z = 0 \quad \text{for } n = 4, 8, 12, \ldots,$$

$$M_z = \pm n\omega^2\left(s_{\frac{n}{2}} \sin\frac{n}{2}\psi - s_{\frac{3n}{2}} \sin\frac{3n}{2}\psi + - \cdots\right) \quad \text{for } n = \begin{cases} 2, 10, 18, \ldots \\ 6, 14, 22, \ldots, \end{cases}$$

$$M_z = \mp n\omega^2(s_n \sin n\psi - s_{3n} \sin 3n\psi + - \cdots) \quad \text{for } n = \begin{cases} 3, 7, 11, \ldots \\ 5, 9, 13, \ldots. \end{cases}$$

(5)

But for $n = 2$, unexpectedly,

$$\left.\begin{aligned}
&X = 2\omega^2(q_1 \cos\psi + q_2 \cos 2\psi - q_4 \cos 4\psi + - \cdots),\\
&Y = 2\omega^2(r_1 \sin\psi + r_2 \sin 2\psi - r_4 \sin 4\psi + - \cdots).
\end{aligned}\right\} \tag{5a}$$

The numerical values of the coefficients α_i and β_i are given in Sect. 11. Moreover, the statements made in Sect. 10 and 11 regarding the longitudinal force X and the rolling moment M_x apply here exactly, provided cylinder-number is replaced by number of cylinders per row. The transverse force Y is no longer balanced automatically, as before, but is described by the same statements as is X.

Likewise, the formula given previously for balancing the longitudinal

force X by means of rotating balance-weights (Sect. 15) can be taken over with certain obvious modifications and used for the elimination of the transverse force Y, for which special rotating masses will generally be needed because $q_j \neq r_j$. Naturally, this is also true for the moments.

26. The In-line Multi-bank Radial Engine.—A radial engine consisting of n banks, each of p cylinders, and in which the cylinders of one bank lie directly ahead of or behind those of another is called here an in-line multi-bank engine. Here, we must distinguish among four possibilities (actually two corresponding pairs of possibilities) according to the value of the angle γ between any two cranks that follow one another in the crank diagram.

(a) *Cyclic-symmetric crank diagram.*—Here

$$\gamma = \frac{2\pi}{n} \quad \text{or} \quad \gamma = \frac{4\pi}{n}.$$

If first we ignore the effect of articulated connecting-rods we need only to combine formulas (19, 4), (19, 5), (19, 7) with the conclusions of Sect. 10 and 11. The results can be put into the form of the equations of Sect. 25, but with the difference that here many of the q_j, r_j and s_j are always zero, so that the corresponding terms in formulas (25, 2) to (25, 5) vanish. The two tables on p. 68 (which include, for comparison, the cases $p = 1$, the usual in-line engine, and $n = 1$, the simple radial engine) give the lowest unbalanced order of the forces X and Y and the rolling moment M_z under the supposition that the forces X^* and Y^* (19, 4) are eliminated by means of crank counterweights and that the always present rolling moment of zero order $-np\omega R^{(p)}$ is not included. When two numbers appear in one box (or one number and a line) the upper one refers to a crank angle $\gamma = 2\pi/n$, the lower to $\gamma = 4\pi/n$: otherwise, the number refers to both cases. The line indicates complete balancing in all orders.

If the effect of articulated connecting-rods is taken into account, the orders listed in the $p = 1$ column are also the lowest unbalanced ones for $p > 1$.

According to the tables, it appears that for an odd number of cylinders np, with respect to forces it is generally better if $n > p$, while the rolling moment depends only on the product np. (The apparently better results if p is even are less important since most engines are four-stroke, requiring p to be odd.) For example, for balancing inertia forces it is better to have seven banks of three cylinders each than three banks of seven cylinders. Exceptions to this rule occur only for quite large numbers of cylinders, e.g. 5×7, 7×11 and 9×11 (and in the trivial cases $n = 1$ or $p = 1$).

The yawing moment M_z and the pitching moment M_y can be studied most easily by combining the table of Sect. 11 with that of 19, removing from the former all orders that are not marked • or ∈ in the latter. (It is assumed that the crank-angle is $\gamma = 2\pi/n$; otherwise Sect. 11b is used.)

Lowest-order unbalanced inertia forces

n \ p	1	2	3	4	5	6	7	8	9	10	11	12
1	1	1	2	—	4	—	6	—	8	—	10	—
2	2 1	—	2	—	4	—	6	—	8	—	10	—
3	6	—	—	—	6	—	6	—	—	—	12	—
4	4 2	—	4 2	—	4	—	8 6	—	8	—	12 10	—
5	10	—	10	—	—	—	20	—	10	—	10	—
6	6	—	—	—	6	—	6	—	—	—	12	—
7	14	—	14	—	14	—	—	—	28	—	56	—
8	8 4	—	8 4	—	16 4	—	8	—	8	—	32 12	—
9	18	—	—	—	36	—	36	—	—	—	54	—
10	10	—	10	—	—	—	20	—	10	—	10	—
11	22	—	22	—	44	—	22	—	44	—	—	—
12	12 6	—	—	—	24 6	—	36 6	—	—	—	12	—

Lowest-order unbalanced rolling moment

n \ p	1	2	3	4	5	6	7	8	9	10	11	12
1	1	—	3	—	5	—	7	—	9	—	11	—
2	— 1	—	— 3	—	— 5	—	— 7	—	— 9	—	— 11	—
3	3	—	3	—	15	—	21	—	9	—	33	—
4	—	—	—	—	—	—	—	—	—	—	—	—
5	5	—	15	—	5	—	35	—	45	—	55	—
6	— 3	—	— 3	—	— 15	—	— 21	—	— 9	—	— 33	—
7	7	—	21	—	35	—	7	—	63	—	77	—
8	—	—	—	—	—	—	—	—	—	—	—	—
9	9	—	9	—	45	—	63	—	9	—	99	—
10	— 5	—	— 15	—	— 5	—	— 35	—	— 45	—	— 55	—
11	11	—	33	—	55	—	77	—	99	—	11	—
12	—	—	—	—	—	—	—	—	—	—	—	—

For example, in a twenty-eight-cylinder engine consisting of four banks, each of seven cylinders, after the balancing of the first-order forces X^* and Y^*, there remains no moment M_z or M_y of sixth or lower order, provided the crank-diagram order (Sect. 9) is 1 2 4 3.

(b) *Crank diagram with one axis of symmetry.*—The following two crank-angles are important:

$$\gamma = \frac{2\pi}{np} \text{ (two-cycle)}, \qquad \gamma = \frac{4\pi}{np} \text{ (four-cycle)}.$$

For these cases, the crank diagram has a symmetric fan-shape (Fig. 51). Engines of this design have particularly good performance since the firing of all np cylinders occurs without overlap (and at equal intervals, naturally)

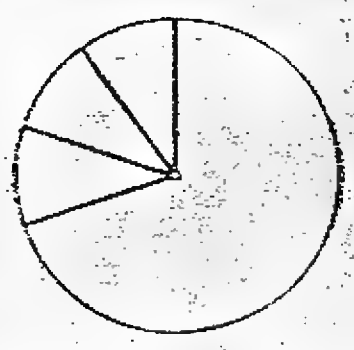


Fig. 51.—Fan-symmetric crank diagram for $n = 4$, $p = 5$, $\gamma = 4\pi/4 \times 5$.

even—in contrast to case (a) above—when n and p have a common factor. The two tables on p. 70, to which the same remarks apply as to the tables opposite, give the lowest orders of the unbalanced forces X and Y and the rolling moment M_x.

The comments made on the previous pair of tables apply equally to this pair. However, in regard to forces, an engine of this type with an odd number of cylinders np is always better if $n < p$ than if $n > p$.

The yawing and pitching moments M_z and M_y must be recalculated. We shall confine our attention to the most important case—the double-bank four-stroke engine ($n = 2$, $\gamma = 2\pi/p$). We designate by X_1, Y_1 the forces in the first bank of cylinders for a crank rotation φ, and by X_2, Y_2 those in the second bank for a crank rotation $\varphi + \gamma$, letting a be the distance between banks. We get

$$M_z = \frac{a}{2}(Y_2 - Y_1), \qquad M_y = \frac{a}{2}(X_1 - X_2).$$

From these relations and from equations (19, 4) and (19, 5) there result, after a simple calculation,

$$\left.\begin{aligned} M_z^* &= \frac{p}{2}\omega^2 a(Q + 2Q^{(p)}) \sin\frac{\pi}{p}\cos\left(\varphi + \frac{\pi}{p}\right), \\ M_y^* &= \frac{p}{2}\omega^2 a(Q + 2Q^{(p)}) \sin\frac{\pi}{p}\sin\left(\varphi + \frac{\pi}{p}\right) \end{aligned}\right\} \quad \text{for } p = 4, 6, 8, \ldots \qquad (1)$$

Lowest-order unbalanced inertia forces

n \ p	1	2	3	4	5	6	7	8	9	10	11	12
1	1	1	2	—	4	—	6	—	8	—	10	—
2	2 1	—	2	—	4	—	6	—	8	—	10	—
3	6	—	2	—	4	—	6	—	8	—	10	—
4	4 2	—	2	—	4	—	6	—	8	—	10	—
5	10	—	2	—	4	—	6	—	8	—	10	—
6	6	—	2	—	4	—	6	—	8	—	10	—
7	14	—	2	—	4	—	6	—	8	—	10	—
8	8 4	—	2	—	4	—	6	—	8	—	10	—
9	18	—	2	—	4	—	6	—	8	—	10	—
10	10	—	2	—	4	—	6	—	8	—	10	—
11	22	—	2	—	4	—	6	—	8	—	10	—
12	12 6	—	2	—	4	—	6	—	8	—	10	—

Lowest-order unbalanced rolling moment

n \ p	1	2	3	4	5	6	7	8	9	10	11	12
1	1	—	3	—	5	—	7	—	9	—	11	—
2	— 1	—	— 3	—	— 5	—	— 7	—	— 9	—	— 11	—
3	3	—	9	—	15	—	21	—	27	—	33	—
4	—	—	—	—	—	—	—	—	—	—	—	—
5	5	—	15	—	25	—	35	—	45	—	55	—
6	— 3	—	— 9	—	— 15	—	— 21	—	— 27	—	— 33	—
7	7	—	21	—	35	—	49	—	63	—	77	—
8	—	—	—	—	—	—	—	—	—	—	—	—
9	9	—	27	—	45	—	63	—	81	—	99	—
10	— 5	—	— 15	—	— 25	—	— 35	—	— 45	—	— 55	—
11	11	—	33	—	55	—	77	—	99	—	121	—
12	—	—	—	—	—	—	—	—	—	—	—	—

and

$$\left.\begin{aligned} M_x = M_x^* \mp \frac{p}{2}\omega^2 aQ\Big[& A_{p-1}\sin\frac{p-1}{p}\pi\cos(p-1)\Big(\psi+\frac{\pi}{p}\Big)+ \\ & + A_{p+1}\sin\frac{p+1}{p}\pi\cos(p+1)\Big(\psi+\frac{\pi}{p}\Big) - A_{3p-1}\sin\frac{3p-1}{p}\pi\cos(3p-1)\Big(\psi+\frac{\pi}{p}\Big) - \\ & - A_{3p+1}\sin\frac{3p+1}{p}\pi\cos(3p+1)\Big(\psi+\frac{\pi}{p}\Big)+-+-\cdots\Big], \\ M_y = M_y^* \mp \frac{p}{2}\omega^2 aQ\Big[& A_{p-1}\sin\frac{p-1}{p}\pi\sin(p-1)\Big(\psi+\frac{\pi}{p}\Big) - \\ & - A_{p+1}\sin\frac{p+1}{p}\pi\sin(p+1)\Big(\psi+\frac{\pi}{p}\Big) - A_{3p-1}\sin\frac{3p-1}{p}\pi\sin(3p-1)\Big(\psi+\frac{\pi}{p}\Big)+ \\ & + A_{3p+1}\sin\frac{3p+1}{p}\pi\sin(3p+1)\Big(\psi+\frac{\pi}{p}\Big)+-+-\cdots\Big] \end{aligned}\right\} \quad (2)$$

$$\text{for } p = \begin{cases} 3, 7, 11, \ldots \\ 5, 9, 13, \ldots \end{cases}$$

The terms M_x^* and M_y^*, which are the only ones if the number of cylinders per bank p is even, can be eliminated by means of counterweights on the cranks, as in Sect. 19.

For articulated connecting-rods, completely analogous expressions for M_x and M_y can be deduced from (24, 3).

27. The Staggered Multi-bank Radial Engine. — Instead of placing cylinders directly behind one another, we can also offset the cylinders of one bank through a certain angle with respect to those of the next bank. An arrangement that gives good inertia balancing, even with articulated connecting-rods, results if both the cranks and the cylinder-banks are offset by the angle $2\pi/n$. Then, both the crank diagram and the diagram of the axes of the n master cylinders as seen from the axial direction form equiangular n-line radial patterns. Fig. 52 illustrates this for the case $n = 3$, $p = 5$; in it the heavy lines indicate the axes of the three master cylinders and also the positions of the three cranks at the instant of coincidence. The other radial lines show the axes of the remaining cylinders which, with the master cylinder axes, are seen to form a symmetrical radial figure with np members, provided n and p have no common factor; otherwise the number of radial lines is smaller.

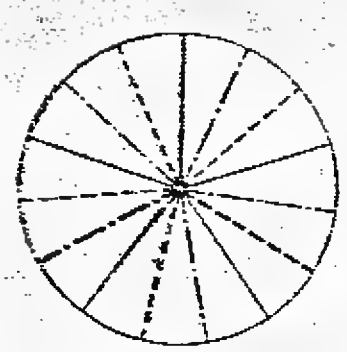


Fig. 52.—Crank and cylinder diagram of multi-bank radial engine with cylinders not in line, for $n = 3$, $p = 5$.

First bank ————;
second bank – – – – –;
third bank – · – · – · – ·

The cyclic symmetry of such an engine is so nearly perfect that it has complete inertia force balance in all orders, even with articulated connecting-rods. This is because the vectors representing the X_k forces of the individual banks of cylinders have equal magnitudes and form a symmetric

n-line radial pattern when projected on to a plane perpendicular to the axis of the engine. Of course, the same holds for Y_k.

The rolling moments, however, combine. From (19, 7) and (24, 3) with $\dot{\omega} = 0$ it follows that:

(a) Connecting-rods attached directly to crankpins:

$$M_z = 0 \quad \text{for } p = 2, 4, 6, \cdots,$$
$$M_z = \mp np\omega^2 R(C_p \sin p\psi - C_{3p} \sin 3p\psi + - \cdots) \quad \text{for } p = \begin{cases} 3, 7, 11, \ldots \\ 5, 9, 13, \ldots \end{cases} \tag{1}$$

(b) Articulated connecting-rods:

$$M_z = n\omega^2 \left\{ R_0 - [1 + (p-1)\varepsilon_1 l_0](R_1 + \varepsilon_2 R_1^*) \right\} \sin \psi. \tag{2}$$

However, these rolling moments are easily eliminated by making $R = 0$ or by using the condition (24, 11) for the first-order rolling moments.

In order to find M_z and M_y also, we must first calculate the moments of the vectors X_k and Y_k which are parallel and perpendicular to the master cylinder axis, expressing the components of these moments according to an (x, y)-system that may be the same as that of the first bank of cylinders, as in Fig. 53. Actually, the corresponding calculation need not be made; the results are better obtained from Sect. 11. For determining the first-order M_z and M_y in this manner, similar force vectors are involved, namely, vectors of equal magnitude which rotate with the several cranks of the normal two-stroke in-line engine. If these vectors are imagined stopped, once for the condition $\psi = 0$ and again for $\psi = 90°$, they will represent the X_k the first time and the Y_k the second time. Hence, referring to (11, 2) and (11, 3) we find

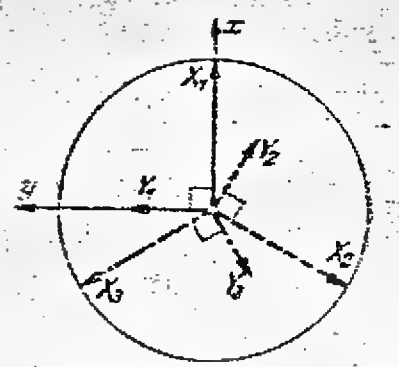


Fig. 53.—Forces X_k, Y_k for $n = 3$ projected on plane normal to axis.

$$M_z = -\alpha_1(X \sin \beta_1 + Y \cos \beta_1), \quad \left.\right\}$$
$$M_y = \alpha_1(X \cos \beta_1 - Y \sin \beta_1). \quad \left.\right\} \tag{3}$$

Here, α_1 and β_1 are the quantities tabulated in Sect. 11, and X and Y are the forces (19, 4), (19, 5) and (22, 8) or (24, 3) so that the numerical calculation of these moments is unnecessary. Note that for $n = 3$ the x-axis is taken in the direction of the axis of the intermediate master cylinder.

If the number of banks n is even, the moments M_z and M_y can be caused to vanish by making the whole engine out of two engines, each with $n/2$ banks of cylinders so arranged as to be mirror images of each other with respect to the symmetry plane normal to the engine axis.

§ 7. Rotary and Counter-rotating Engines

28. The Rotary Engine.—In the rotary engine[1] (Fig. 54) the "crank" OA is fixed while the cylinders, etc., rotate about the axis O. The connecting-rods AB turn about the "crankpin" A. In addition to this rotation about O the pistons oscillate along the line OB. We shall use the symbols already defined, adding ϑ, the inclination of the connecting-rod. For simplicity, we assume that the centre of gravity of each piston coincides with B, the axis of its gudgeon-pin. (Otherwise, the formulas would be considerably more complicated without giving much more information.)

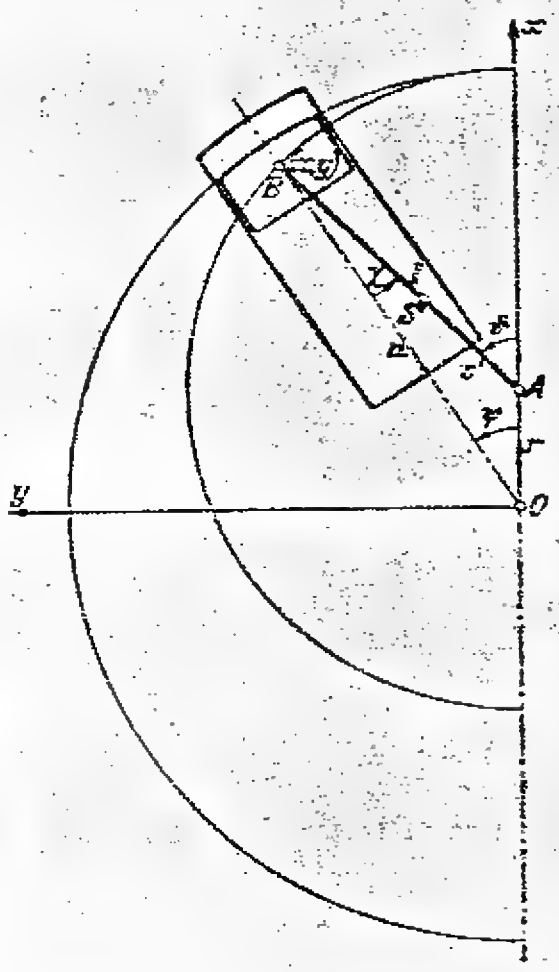


Fig. 54.—Reciprocating mechanism of rotary engine.

Since, for the inertia forces, we may divide the mass of the connecting-rod between the points A and B, we have to deal with only one moving mass, Q/r at B, where Q is defined in (3, 2). Therefore, if x, y are the coordinates of B there results simply

$$X = -Q\frac{\ddot{x}}{r}, \qquad Y = -Q\frac{\ddot{y}}{r}. \tag{1}$$

[1] H. Schrön, "Vergleich der Umlaufmotoren erster und zweiter Art auf Grund von Massenwirkungen", *Motorwagen* 23 (1920) p. 689.

But $x = r + l\cos\vartheta$ and $y = l\sin\vartheta$. If we introduce $\vartheta = \psi + \chi$ and then, as in Sect. 4, use

$$\sin\chi = \lambda\sin\psi,$$
$$\cos\chi = \lambda\left(A_0 + \frac{1}{4}A_2\cos 2\psi - \frac{1}{16}A_4\cos 4\psi + \frac{1}{36}A_6\cos 6\psi - + \cdots\right)$$

we get, after trigonometric manipulation,

$$\left.\begin{aligned}\frac{x}{r} &= \frac{1}{2} + \left(A_0 + \frac{1}{8}A_2\right)\cos\psi + \frac{1}{2}\cos 2\psi + \frac{1}{2}\left(\frac{A_2}{4} - \frac{A_4}{16}\right)\cos 3\psi - \frac{1}{2}\left(\frac{A_4}{16} - \frac{A_6}{36}\right)\cos 5\psi + - \cdots,\\ \frac{y}{r} &= \left(A_0 - \frac{1}{8}A_2\right)\sin\psi + \frac{1}{2}\sin 2\psi + \frac{1}{2}\left(\frac{A_2}{4} + \frac{A_4}{16}\right)\sin 3\psi - \frac{1}{2}\left(\frac{A_4}{16} + \frac{A_6}{36}\right)\sin 5\psi + - \cdots.\end{aligned}\right\} \quad (2)$$

Repeated differentiation with respect to time gives

$$\left.\begin{aligned}-\frac{\ddot{x}}{r} &= \omega^2\left[\left(A_0 + \frac{1}{8}A_2\right)\cos\psi + 2\cos 2\psi + \frac{9}{2}\left(\frac{A_2}{4} - \frac{A_4}{16}\right)\cos 3\psi - \right.\\ &\qquad \left. - \frac{25}{2}\left(\frac{A_4}{16} - \frac{A_6}{36}\right)\cos 5\psi + - \cdots\right] + ((\dot{\omega})),\\ -\frac{\ddot{y}}{r} &= \omega^2\left[\left(A_0 - \frac{1}{8}A_2\right)\sin\psi + 2\sin 2\psi + \frac{9}{2}\left(\frac{A_2}{4} + \frac{A_4}{16}\right)\sin 3\psi - \right.\\ &\qquad \left. - \frac{25}{2}\left(\frac{A_4}{16} + \frac{A_6}{36}\right)\sin 5\psi + - \cdots\right] + ((\dot{\omega})).\end{aligned}\right\} \quad (3)$$

where the symbol $((\dot{\omega}))$ has the same significance as in Sect. 5. These expressions are to be inserted in (1).

We shall write down directly the result for a rotary engine of p cylinders whose axes make equal angles with each other, so that the position of the ith cylinder at any time is

$$\psi_i = \psi + (i-1)\frac{2\pi}{p} \qquad (i = 1, 2, \ldots, p).$$

The resulting summations $\Sigma\cos j\psi_i$ and $\Sigma\sin j\psi_i$, which are taken from $i = 1$ to $i = p$, have been evaluated in Sect. 10. Therefore, the inertia forces are as follows:

$$\left.\begin{aligned}X &= 4\omega^2 Q\cos 2\psi + ((\dot{\omega}))\\ Y &= 4\omega^2 Q\sin 2\psi + ((\dot{\omega}))\end{aligned}\right\} \text{ for } p = 2,$$

$$\left.\begin{aligned}X &= 0\\ Y &= 0\end{aligned}\right\} \text{ for } p = 4, 6, 8, \ldots,$$

$$\left.\begin{aligned}X &= \pm\frac{p^2}{2}\omega^2 Q\left\{\left[\frac{A_{p-1}}{(p-1)^2} - \frac{A_{p+1}}{(p+1)^2}\right]\cos p\psi - \right.\\ &\qquad \left. - 3^2\left[\frac{A_{2p-1}}{(3p-1)^2} - \frac{A_{2p+1}}{(3p+1)^2}\right]\cos 3p\psi + - \cdots\right\} \pm ((\dot{\omega})),\\ Y &= \pm\frac{p^2}{2}\omega^2 Q\left\{\left[\frac{A_{p-1}}{(p-1)^2} + \frac{A_{p+1}}{(p+1)^2}\right]\sin p\psi - \right.\\ &\qquad \left. - 3^2\left[\frac{A_{2p-1}}{(3p-1)^2} + \frac{A_{2p+1}}{(3p+1)^2}\right]\sin 3p\psi + - \cdots\right\} \pm ((\dot{\omega}))\end{aligned}\right\} \text{ for } p = \begin{cases}3, 7, 11, \ldots\\ 5, 9, 13, \ldots\end{cases} \quad (4)$$

Next, to get the rolling moment M_z we consider the two equivalent masses of each connecting-rod, together with the corresponding radii of gyration obtained in Sect. 2. Then the angular momentum of a connecting-rod with respect to O is made up of two parts $(k^2 G_1 + k_2^2 G_2)\dot{\vartheta}/g \equiv -R\dot{\vartheta}/\lambda$, and $G_2 u^2 \omega/g$, where u is the distance OB. If k'' is the radius of gyration of the piston relative to an axis through its centre of gravity B and parallel to the rotation axis O, its angular momentum relative to O is $G''(k''^2 + u^2)\omega/g$. Finally, the angular momentum of a rotating cylinder is $D\omega$, where D is its moment of inertia relative to O. Then

$$M_z = R\frac{\dot{\delta}}{\lambda} - \frac{Q}{\tau}\frac{d}{dt}(u^2\omega) - \left(\frac{G''}{g}k''^2 + D\right)\dot{\omega} \tag{5}$$

But from Fig. 54 we get

$$u^2 = l^2 + r^2 + 2lr\cos\vartheta = l^2 + r^2 + 2r(x - r)$$

or

$$\frac{u^2}{r^2} = \frac{1}{\lambda^2} - 1 + 2\frac{x}{r}.$$

Hence, the rolling moment (5) becomes

$$M_z = R\frac{\dot{\delta}}{\lambda} - 2rQ\frac{\omega\dot{x}}{r} - \dot{\omega}\left[rQ\left(\frac{1}{\lambda^2} - 1 + 2\frac{x}{r}\right) + \frac{G''}{g}k''^2 + D\right]. \tag{6}$$

Here, $\dot{\delta} = \dot{\omega} + \ddot{\chi}$, or with the value of $\ddot{\chi}$ from (4, 11), which is still correct,

$$-\frac{\dot{\delta}}{\lambda} = \omega^2(C_1 \sin\psi - C_3 \sin 3\psi + - \cdots) + ((\dot{\omega})) - \frac{\dot{\omega}}{\lambda}. \tag{7}$$

From (2),

$$\left.\begin{aligned} -\frac{\omega\dot{x}}{r} = \omega^2\Big[\Big(A_1 + \frac{1}{8}A_3\Big)\sin\psi + \sin 2\psi + \frac{3}{2}\Big(\frac{A_3}{4} - \frac{A_5}{16}\Big)\sin 3\psi - \\ - \frac{5}{2}\Big(\frac{A_5}{16} - \frac{A_7}{36}\Big)\sin 5\psi + - \cdots\Big]. \end{aligned}\right\} \tag{8}$$

These expressions must be inserted into (6).

We write down again the results for the symmetrical rotary engine with p cylinders, omitting terms containing $\dot{\omega}$, which are comparatively unimportant in the rotary engine:

$$\left.\begin{aligned} &M_z = 4\omega^2 rQ\sin 2\psi \quad \text{for } p = 2, \\ &M_z = 0 \quad \text{for } p = 4, 6, 8, \ldots, \\ &M_z = \mp p\omega^2\Big[\Big[prQ\Big(\frac{A_{p-1}}{(p-1)^2} - \frac{A_{p+1}}{(p+1)^2}\Big) + RC_p\Big]\sin p\psi - + \cdots\Big] \text{ for } p = \begin{cases} 3, 7, 11, \ldots \\ 5, 9, 13, \ldots \end{cases} \end{aligned}\right\} \tag{9}$$

From (4) and (9) we draw the following conclusions:

(1) The rotary engine with even numbers of cylinders $p > 2$ is completely balanced with respect to forces and moments; with an odd number

of cylinders the lowest-order unbalanced forces and moments are both of order p.

(2) If we need not be concerned with forces and rolling moments of sixth and higher order, we may consider engines with more than six cylinders to be balanced.

These statements must be changed somewhat when engines with articulated connecting-rods are in question, as in Sect. 21 to 24.

The orders of the unbalanced forces and rolling moments are indicated in the adjacent table.

Unbalanced forces ● and steady rolling moments ●

No. cyl. \ Order	1	2	3	4	5	6	7	8	9	10	11	12
2		●										
3			●						●			
4												
5					●							
6												
7							●					
8												
9									●			
10												
11											●	
12												

The unbalanced steady force of lowest order is a vector rotating about O. If $p = 2$ the length of this vector is constant, describing a circle, while if $p > 2$ it describes an ellipse. The extreme values of the length are

$$4\omega^2 Q \quad \text{for} \quad p = 2 \quad \text{and} \quad \frac{p^2}{2}\omega^2 Q\left[\frac{A_{p-1}}{(p-1)^2} \mp \frac{A_{p+1}}{(p+1)^2}\right] \quad \text{for } p = 3, 5, 7, \ldots .$$

This vector (if $p = 2$) or its eccentric anomaly (if $p > 2$) rotates with an angular velocity $p\omega$ so that if $p = 2, 3, 7, 11, \ldots$ it passes the positive x-axis at the instant when any cylinder axis also passes it; if $p = 5, 9, 13, \ldots$ it coincides with the negative x-axis whenever a cylinder axis passes the positive x-axis.

Unfortunately, the rotary engine with an even number of cylinders is only suitable for the two-stroke cycle. The more important four-stroke engine requires an odd number of cylinders, which is always less satisfactory for balancing.

29. The Counter-rotating Engine. — The large inertia forces and the

generally even more troublesome gyroscopic moment of the rotary engine can be cut down to about one-fourth by reducing the angular velocity ω of the cylinders by one-half and letting the crank rotate at the same speed in the opposite direction. The result is the counter-rotating engine in which the angular velocity of the cylinders ω' and the opposite angular velocity ω of the crank are kept in a fixed relation by means of gearing, so that with the factor $\varkappa$, which is normally unity,

$$\omega' = \varkappa\omega. \tag{1}$$

For the corresponding angles of rotation (Fig. 55),

$$\psi' = \varkappa\psi + \delta, \tag{2}$$

in which δ is a constant angle whose value depends on the choice of reference

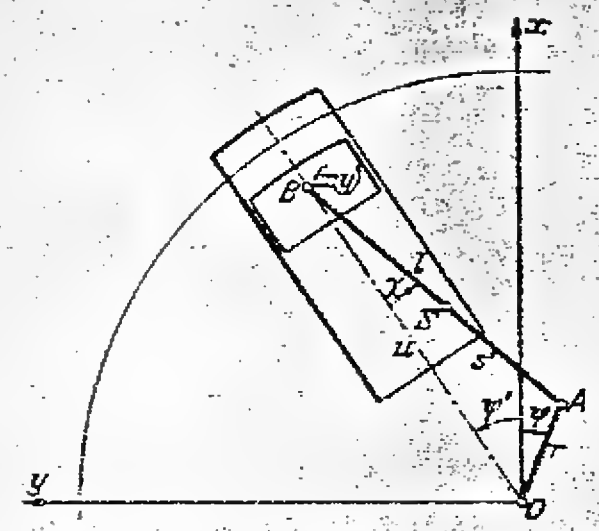


Fig. 55.—Reciprocating mechanism of counter-rotating engine.

axes for ψ and ψ'. (In Fig. 55 both are measured from the x-axis, which is arbitrarily made vertical.)

If again the centre of gravity of the piston is assumed to be on the axis of the gudgeon-pin, the inertia forces become (as in (3, 3) and (28, 1))

$$\left.\begin{aligned} X &= Q'(\omega^2\cos\psi + \dot{\omega}\sin\psi) - Q\frac{\ddot{x}}{g}, \\ Y &= -Q'(\omega^2\sin\psi - \dot{\omega}\cos\psi) - Q\frac{\ddot{y}}{g}. \end{aligned}\right\} \tag{3}$$

But

$$x = r\cos\psi + l\cos(\psi' + \chi), \qquad y = -r\sin\psi + l\sin(\psi' + \chi).$$

On introducing

$$\sin\chi = \lambda\sin(\psi + \psi'),$$

$$\cos\chi = \lambda\left[A_0 + \frac{1}{4}A_2\cos 2(\psi + \psi') - \frac{1}{16}A_4\cos 4(\psi + \psi') + -\cdots\right]$$

we get the general expressions

$$\left.\begin{aligned}\frac{x}{r} &= \cos\psi - \sin\psi'\sin(\psi+\psi') + \cos\psi'\left[A_0 + \frac{1}{4}A_2\cos 2(\psi+\psi') - \right.\\ &\qquad \left. - \frac{1}{16}A_4\cos 4(\psi+\psi') + - \cdots\right],\\ \frac{y}{r} &= -\sin\psi + \cos\psi'\sin(\psi+\psi') + \sin\psi'\left[A_0 + \frac{1}{4}A_2\cos 2(\psi+\psi') - \right.\\ &\qquad \left. - \frac{1}{16}A_4\cos 4(\psi+\psi') + - \cdots\right].\end{aligned}\right\} \tag{4}$$

From now on we confine ourselves to the most important practical case, $\varkappa = 1$, so that $\omega' = \omega$, and the crank and the cylinders rotate oppositely at the same speed. On introducing (2) we get

$$\left.\begin{aligned}\frac{x}{r} &= \frac{1}{2}\cos\psi + \frac{1}{2}\cos(3\psi+2\delta) + A_0\cos(\psi+\delta) + \\ &\quad + \frac{1}{2}\frac{A_2}{4}[\cos(3\psi+\delta) + \cos(5\psi+3\delta)] - \\ &\quad - \frac{1}{2}\frac{A_4}{16}[\cos(7\psi+3\delta) + \cos(9\psi+5\delta)] + - \cdots,\\ \frac{y}{r} &= -\frac{1}{2}\sin\psi + \frac{1}{2}\sin(3\psi+2\delta) + A_0\sin(\psi+\delta) - \\ &\quad - \frac{1}{2}\frac{A_2}{4}[\sin(3\psi+\delta) - \sin(5\psi+3\delta)] + \\ &\quad + \frac{1}{2}\frac{A_4}{16}[\sin(7\psi+3\delta) - \sin(9\psi+5\delta)] - + \cdots.\end{aligned}\right\} \tag{5}$$

Repeated differentiation with respect to time gives

$$\left.\begin{aligned}-\frac{\ddot{x}}{r} &= \omega^2\left\{\frac{1}{2}\cos\psi + \frac{9}{2}\cos(3\psi+2\delta) + A_0\cos(\psi+\delta) - \right.\\ &\quad + \frac{1}{2}\frac{A_2}{4}[9\cos(3\psi+\delta) + 25\cos(5\psi+3\delta)] - \\ &\quad \left. - \frac{1}{2}\frac{A_4}{16}[49\cos(7\psi+3\delta) + 81\cos(9\psi+5\delta)] + - \cdots\right\} + (i\omega),\\ -\frac{\ddot{y}}{r} &= \omega^2\left\{-\frac{1}{2}\sin\psi + \frac{9}{2}\sin(3\psi+2\delta) + A_0\sin(\psi+\delta) - \right.\\ &\quad - \frac{1}{2}\frac{A_2}{4}[9\sin(3\psi+\delta) - 25\sin(5\psi+3\delta)] + \\ &\quad \left. + \frac{1}{2}\frac{A_4}{16}[49\sin(7\psi+3\delta) - 81\sin(9\psi+5\delta)] - + \cdots\right\} + (i\omega).\end{aligned}\right\} \tag{6}$$

These expressions are to be put into equation (3).

We shall write down the results only for the case of a rotationally symmetric counter-rotating engine with p cylinders. For the ith cylinder

$$\delta_i = (i-1)\frac{2\pi}{p} \qquad (i = 1, 2, \ldots, p).$$

The subscript I designates the cylinder whose axis passes the positive x-axis simultaneously with the crank. By summation, we get the following values for the inertia forces:

$$\left.\begin{aligned} X &= \omega^2[(Q+2Q'')\cos\psi + 9Q\cos 3\psi] + ((\omega)) \\ Y &= \omega^2[-(Q+2Q'')\sin\psi + 9Q\sin 3\psi] + ((\omega)) \end{aligned}\right\} \text{ for } p=2,$$

$$\left.\begin{aligned} X &= \frac{p}{2}\omega^2(Q+2Q^{(p)})\cos\psi + ((\omega)) \\ Y &= -\frac{p}{2}\omega^2(Q+2Q^{(p)})\sin\psi + ((\omega)) \end{aligned}\right\} \text{ for } p=4,6,8,\ldots,$$

$$\left.\begin{aligned} X &= \frac{p}{2}\omega^2\left\{(Q+2Q^{(p)})\cos\psi \pm Q\left[\left(\frac{2p-1}{p-1}\right)^2 A_{p-1}\cos(2p-1)\psi - \right.\right. \\ &\qquad \left.\left. -\left(\frac{2p+1}{p+1}\right)^2 A_{p+1}\cos(2p+1)\psi - + + - \cdots\right]\right\} + ((\omega)) \\ Y &= \frac{p}{2}\omega^2\left\{-(Q+2Q^{(p)})\sin\psi \pm Q\left[\left(\frac{2p-1}{p-1}\right)^2 A_{p-1}\sin(2p-1)\psi + \right.\right. \\ &\qquad \left.\left. +\left(\frac{2p+1}{p+1}\right)^2 A_{p+1}\sin(2p+1)\psi - - + + \cdots\right]\right\} + ((\omega)) \end{aligned}\right\} \text{ for } p=\begin{cases}3,7,11,\ldots\\5,9,13,\ldots\end{cases} \quad (7)$$

$Q^{(p)}$ is the same symbol that was used in connection with fan-like and radial engines (18, 7). In the present case, as before, the first-order inertia force can be eliminated by use of a crank counterweight which satisfies the condition $Q + 2Q^{(p)} = 0$, which is given in (18, 8).

Analogous to (28, 5), the rolling moment is

$$M_z = R'\omega + R\frac{\omega' + \ddot{\chi}}{\lambda} - \frac{Q}{r}\frac{d}{dt}(x^2\omega') - \left(\frac{G''}{g}k''^2 + D\right)\dot{\omega}'. \quad (8)$$

In D are to be included the moment of inertia of the cylinder and the equivalent moment of inertia (in terms of ω') of the gearing. The rolling moment can also be expressed in the form

$$M_z = R\frac{\ddot{\chi}}{\lambda} - \omega' rQ\frac{d}{dt}\left(\frac{u}{r}\right)^2 + \omega R' - \dot{\omega}'\left[rQ\left(\frac{x}{r}\right)^2 + \frac{G''}{g}k''^2 + D - \frac{R}{\lambda}\right]. \quad (9)$$

Here we find, in the same manner as in (4, 11),

$$-\frac{\ddot{\chi}}{\lambda} = (\omega+\omega')^2[C_1\sin(\psi+\psi') - C_3\sin 3(\psi+\psi') + - \cdots] + ((\omega+\omega')). \quad (10)$$

Finally, from Fig. 55,

$$\begin{aligned} u^2 &= l^2 + r^2 + 2lr\cos(\psi+\psi'+\chi) \\ &= l^2 + r^2 + 2r[(x - r\cos\psi)\cos\psi - (y + r\sin\psi)\sin\psi] \end{aligned}$$

or

$$\left(\frac{u}{r}\right)^2 = \frac{l^2}{r^2} - 1 + 2\left(\frac{x}{r}\cos\psi - \frac{y}{r}\sin\psi\right). \quad (11)$$

M_z can be calculated in general form from equations (9), (10), (11) and (5).

In the special case where $\varkappa = 1$, or $\omega' = \omega$, to which we have limited

ourselves, it follows from (2) and (5) that

$$M_z = R\frac{\ddot{\chi}}{\lambda} - \omega r Q \frac{d}{dt}\left(\frac{u}{r}\right)^2 - \omega\left[rQ\left(\frac{u}{r}\right)^2 + \frac{G''}{g}k''^2 + D - \frac{R}{\lambda} - R'\right], \tag{12}$$

with

$$-\frac{\ddot{\chi}}{\lambda} = 4\omega^2[C_1 \sin(2\psi + \delta) - C_3 \sin(6\psi + 3\delta) + - \cdots] + ((\dot{\omega})), \tag{13}$$

$$\left.\begin{aligned}\left(\frac{u}{r}\right)^2 = \frac{1}{\lambda^2} + \cos(4\psi + 2\delta) + \left(2A_0 + \frac{1}{4}A_2\right)\cos(2\psi + \delta) + \left(\frac{A_2}{4} - \frac{A_4}{16}\right)\cos(6\psi + 3\delta) - \\ - \left(\frac{A_4}{16} - \frac{A_6}{36}\right)\cos(10\psi + 5\delta) + - \cdots\end{aligned}\right\} \tag{14}$$

and

$$\left.\begin{aligned}-\frac{d}{dt}\left(\frac{u}{r}\right)^2 = \omega\left[4\sin(4\psi + 2\delta) + \left(4A_0 + \frac{1}{2}A_2\right)\sin(2\psi + \delta) + \right. \\ \left. + 6\left(\frac{A_2}{4} - \frac{A_4}{16}\right)\sin(6\psi + 3\delta) - 10\left(\frac{A_4}{16} - \frac{A_6}{36}\right)\sin(10\psi + 5\delta) + - \cdots\right].\end{aligned}\right\} \tag{15}$$

For a rotationally symmetric counter-rotating engine with p cylinders we get by summation, omitting the terms with $\dot{\omega}$,

$$\left.\begin{aligned}&M_z = 8\omega^2 r Q \sin 4\psi \quad \text{for } p = 2, \\ &M_z = 0 \quad \text{for } p = 4, 6, 8, \ldots, \\ &M_z = \pm 2p\omega^2\left\{\left[prQ\left(\frac{A_{p-1}}{(p-1)^2} - \frac{A_{p+1}}{(p+1)^2}\right) + 2BC_p\right]\sin 2p\psi - + \ldots\right\} \text{for } p = \begin{cases}3, 7, 11, \ldots \\ 5, 9, 13, \ldots\end{cases}\end{aligned}\right\} \tag{16}$$

From (7) and (16) we draw the following conclusions:

(1) Provided the first-order forces are eliminated by use of a counter-weight on the crank, the counter-rotating engine with an even number of cylinders $p > 2$ is entirely balanced with respect to forces and rolling moment; on the other hand, with an odd number of cylinders the lowest-order unbalanced force is of order $2p - 1$, and the lowest-order unbalanced rolling moment is of order $2p$.

(2) If forces and moments of fifth and higher order need not be considered, engines with three or more cylinders may be considered balanced. (With articulated connecting-rods these statements require corresponding modification.)

Unbalanced forces ● ⊕ and steady rolling moments ○

No. cyl. \ Order	1	2	3	4	5	6	7	8	9	10	11	12
2	⊕		●	○								
3	⊕				●	○	●					
4	⊕											
5	⊕								●	○	●	
6	⊕											
7	⊕											

The table opposite shows the orders of the unbalanced forces and rolling moments. (The forces marked ⊕ can be balanced. For $p > 5$ all unbalance is of order $j > 12$.) Comparison with Sect. 28 shows that the counter-rotating engine has much better inertia balance than the rotary engine, provided the rather awkward crank counterweight is accepted.

The lowest-order unbalanced steady force is a vector of magnitude

$$\frac{p^2(2p-1)^2}{2(p-1)^2}\omega^2 Q A_{p-1} \quad (\text{with } A_1 \equiv 1).$$

This vector rotates with angular velocity $(2p-1)\omega$ so that it overtakes the crank at the instant when the crank passes the upper dead centre of each cylinder, provided $p = 2, 3, 7, 11, \ldots$ or overtakes its backward extension if $p = 5, 9, 13, \ldots$.

PART II

Power Smoothing

1. Introduction.—A second problem confronts the internal-combustion engine designer as soon as he attempts to follow the flow of energy from the burning gases in the cylinder to the load. Three sets of questions are involved: How large is the useful torque in the crankshaft derived from the gases in the cylinders? What forces are produced in the various elements of the mechanism by this flow of energy? How large an energy reservoir (perhaps in the form of a flywheel) is necessary to ensure a prescribed degree of smoothness in the motion? (An alternative to this last is: What kind of motion results in the case of a particular energy reservoir?) At the same time, it is necessary to investigate the friction forces in the engine and the consequent losses of energy. Since we may consider the power output of an engine just as well as the flow of energy, and since every engine must produce its power at as constant a rate as possible, we are justified in grouping these questions together under the term "power smoothing".

We shall first make a thorough study of the one-cylinder engine (§ 1); these results can be extended easily and quickly to in-line, multi-row, radial, rotary and counter-rotating engines (§ 2).

§ 1. The One-cylinder Engine

2. Its Power Balance.—We begin by determining the power balance of the one-cylinder engine, assumed first to be single-acting. Let p be the instantaneous (gauge) pressure in the cylinder on the combustion side (becoming negative at times), F the cross-sectional area of the piston, so that $K = pF$ is the force of the burning gases on the piston, and, finally, v the velocity of the piston, positive in the direction of K. Then Kv is the instantaneous power which the burning gases supply to the engine. Next, let $\mathbf{M}^{(s)}$ be the useful torque taken from the crankshaft whose angular velocity is ω, so that $\mathbf{M}^{(s)}\omega$ is the useful power of the engine. (Note that a bold-face $\mathbf{M}$ symbolizes torque or turning moment, in order to avoid confusion with other kinds of moment for which italicized M is used.) Finally, let T represent the total kinetic energy of the mechanism (piston, gudgeon-pin, crosshead, connecting-rod, crank, counterweights and shaft), U its potential

energy due to gravity, and V the power lost through friction. Then the power balance in the engine is given by the expression

$$Kv - \mathbf{M}^{(n)}\omega - V = \frac{d}{dt}(T + U). \tag{1}$$

The useful torque is

$$\mathbf{M}^{(n)} = K\frac{v}{\omega} - \frac{1}{\omega}\left(\frac{dT}{dt} + \frac{dU}{dt} + V\right). \tag{2}$$

Since T and U are functions of the crank rotation φ and since $d\varphi = \omega\, dt$, (2) can be replaced by

$$\mathbf{M}^{(n)} = K\frac{v}{\omega} - \left(\frac{dT}{d\varphi} + \frac{dU}{d\varphi} + \frac{V}{\omega}\right). \tag{3}$$

In low-speed engines the first term on the right-hand side outweighs the others; in high-speed engines, on the other hand, we may not ignore the term with T. We now proceed to examine the individual terms of the equation.

3. The Torque of the Equivalent Tangential Force.—In order to evaluate the first term on the right-hand side of equation (3) we make use of Fig. 1, in which is sketched a standard reciprocating mechanism. In terms of the symbols defined there—v the piston speed, $r\omega$ the velocity of A (the crankpin axis), and M the instantaneous centre of the connecting-rod motion—we find

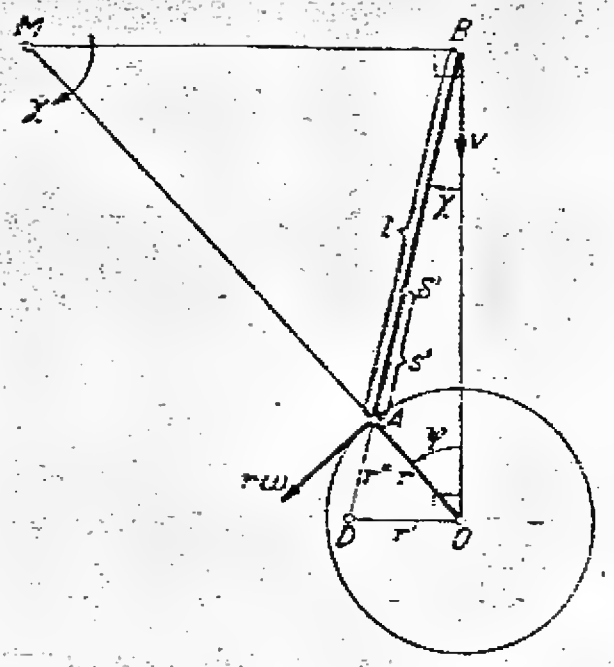


Fig. 1.—Standard reciprocating mechanism (without offset).

$$\frac{v}{r\omega} = \frac{MB}{MA} = \frac{r'}{r}.$$

Therefore

$$\mathbf{M}' \equiv K\frac{r'}{\omega} = Kr'. \tag{1}$$

Here r' is the length OD (positive to the left) defined by the intersection of a horizontal line from O and the axis AB of the connecting-rod (extended if necessary). We then write

$$\left.\begin{aligned} \mathbf{M}' &= Pr \\ \text{with}\quad P &= K\frac{r'}{r}, \end{aligned}\right| \tag{2}$$

where P can be thought of as a force at the crankpin tangent to the crankpin circle, which furnishes the same power as the piston force K. P is called the equivalent tangential force and must not be confused with the actual force on the crankpin, with which it corresponds only for weightless, frictionless mechanisms, and which is calculated in Sect. 6. We call $\mathbf{M}'$ the torque of the equivalent tangential force.

The piston force is generally given by means of an indicator diagram as function of the piston travel. Fig. 2 shows how the equivalent tangential force or the tangential-force torque can be found from the indicator diagram for a given crank rotation ψ. A semicircle is first drawn with the total piston travel as diameter; the choice of ψ determines A, which fixes B since $AB = l$, the distance between centres of crankpin and cross-head or gudgeon-pin. Next the arc AC is drawn with B as centre. Then, directly above C the corresponding piston force K is read off and multiplied by $OD = r'$ in order to get $\mathfrak{M}'$. Alternatively, the value of K is laid off along OA, locating E, from which EF is drawn parallel to BA, giving $OF = P$ to the same scale

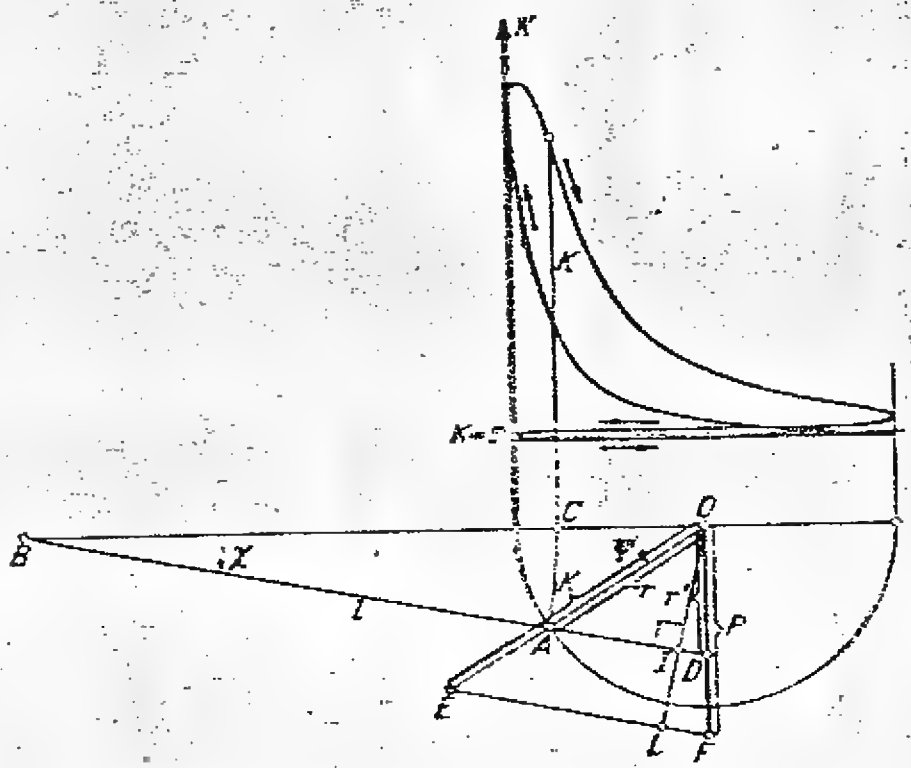


Fig. 2.—Determination of the tangential crank force P and its moment $\mathfrak{M}'$ from the piston force K.

as is used for $K = OE$. The semicircle shown is used for the entire cycle; for $\pi < \psi < 2\pi$ the crank rotation is found by reflection with respect to the diameter of the semicircle; r' is negative and the sign of OF is opposite that of K. The results are plotted against ψ as abscissa between $\psi = 0$ and $\psi = 2\pi$ in two-stroke cycle engines and between $\psi = 0$ and $\psi = 4\pi$ in the four-stroke cycle as shown in Figs. 3 and 4. The ordinates represent magnitudes of the equivalent tangential force P or its torque $\mathfrak{M}' = Pr$, and are zero at all dead-centres $\psi = 0, \pi, 2\pi, \ldots$.

In double-acting engines we may construct the separate equivalent tangential-force curves corresponding to the two indicator diagrams. As shown by the dashed and dotted curves of Fig. 5, even with identical indicator diagrams the tangential-force curves will differ from each other, depending on the magnitude of the ratio r/l. Finally, by adding ordinates we get the resultant tangential-force curve shown by the solid line of Fig. 5. Alter-

natively, the indicator diagrams can be combined to give the pressure difference on the two ends of the piston, from which the equivalent tangential-force curve is found directly.

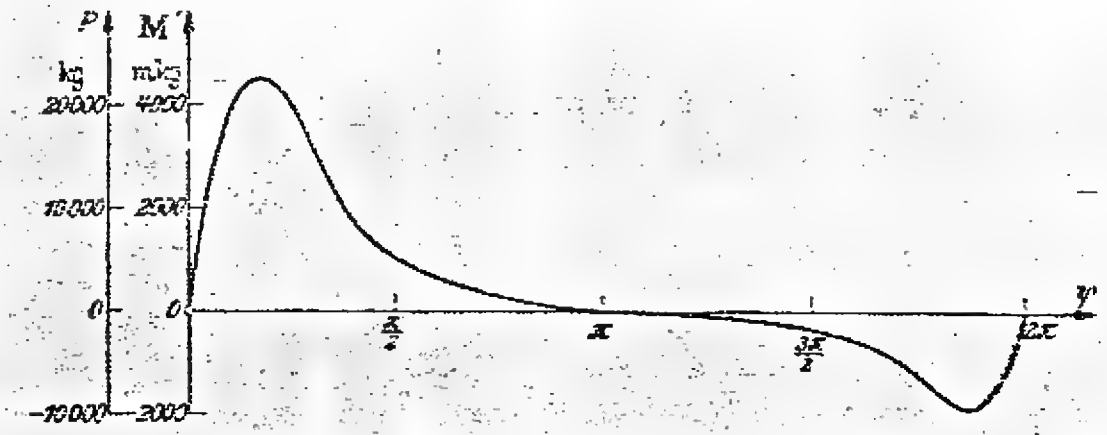


Fig. 3.—Tangential crank force for two-cycle engine.

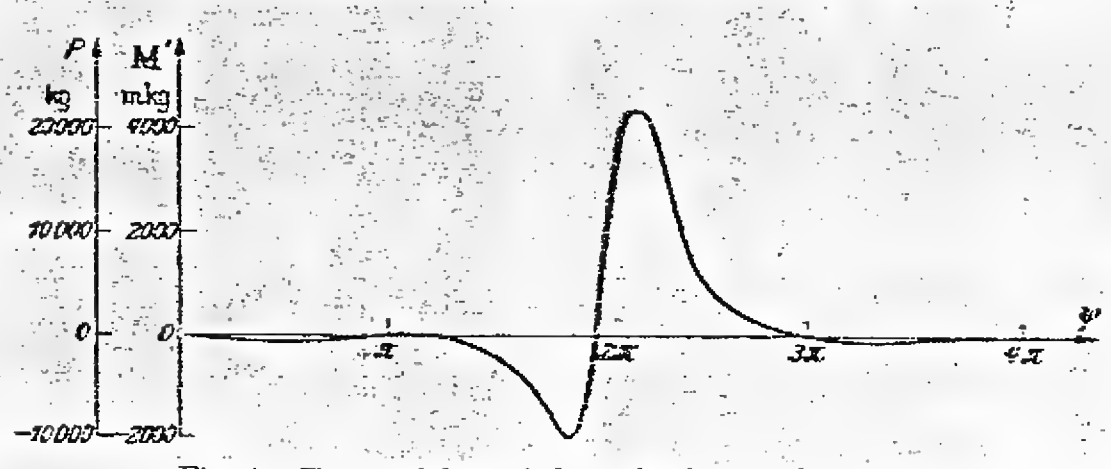


Fig. 4.—Tangential crank force for four-cycle engine.

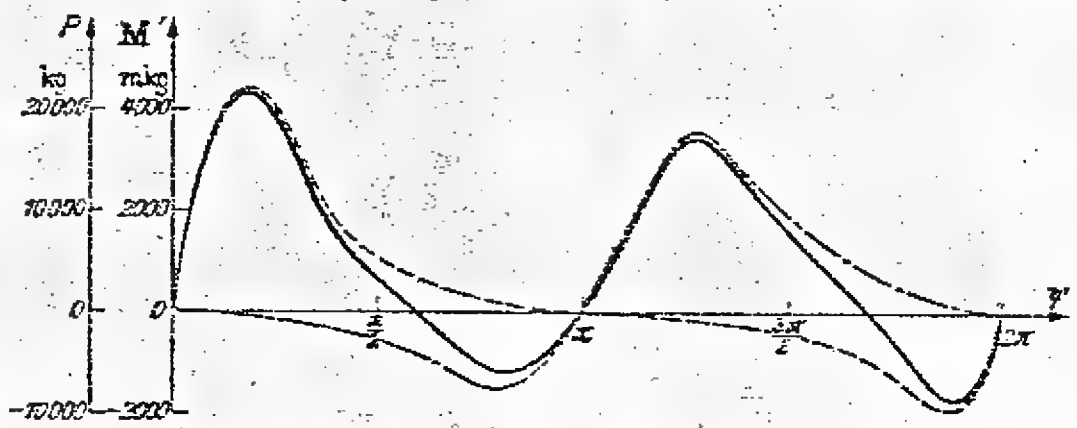


Fig. 5.—Tangential crank force for double-acting two-cycle engine.

Later, it will be found useful to make a harmonic analysis of the tangential-force torque (Vol. I, Part III, Sect. 3 to 5). The result is a Fourier series of one of the following forms:

(a) Two-stroke cycle engines

$$M' = M_i\left(1 + \sum_{j=1}^{\infty} a_j \cos j\psi + \sum_{j=1}^{\infty} b_j \sin j\psi\right). \tag{3}$$

(b) Four-stroke cycle engines

$$M' = M_i\left(1 + \sum_{j=1}^{\infty} a_j \cos j\frac{\psi}{2} + \sum_{j=1}^{\infty} b_j \sin j\frac{\psi}{2}\right) \tag{4}$$

Here M_0' is the mean torque of the tangential force. In the four-stroke engine we shall define the order j of a term on the basis of two crank revolutions per period. In engineering practice it is customary to take the period as the time required for a single revolution, making the order numbers $j = \frac{1}{2}, 1, 1\frac{1}{2}, 2, 2\frac{1}{2}$, etc., instead of $j = 1, 2, 3, 4, 5$, etc., as will be done here. Fig. 6 shows the Fourier coefficients a_j and b_j for the four-stroke engine of Fig. 4, as well as the Fourier amplitudes

$$c_j = \sqrt{a_j^2 + b_j^2}. \tag{5}$$

It is apparent that rather high orders can have considerable importance.

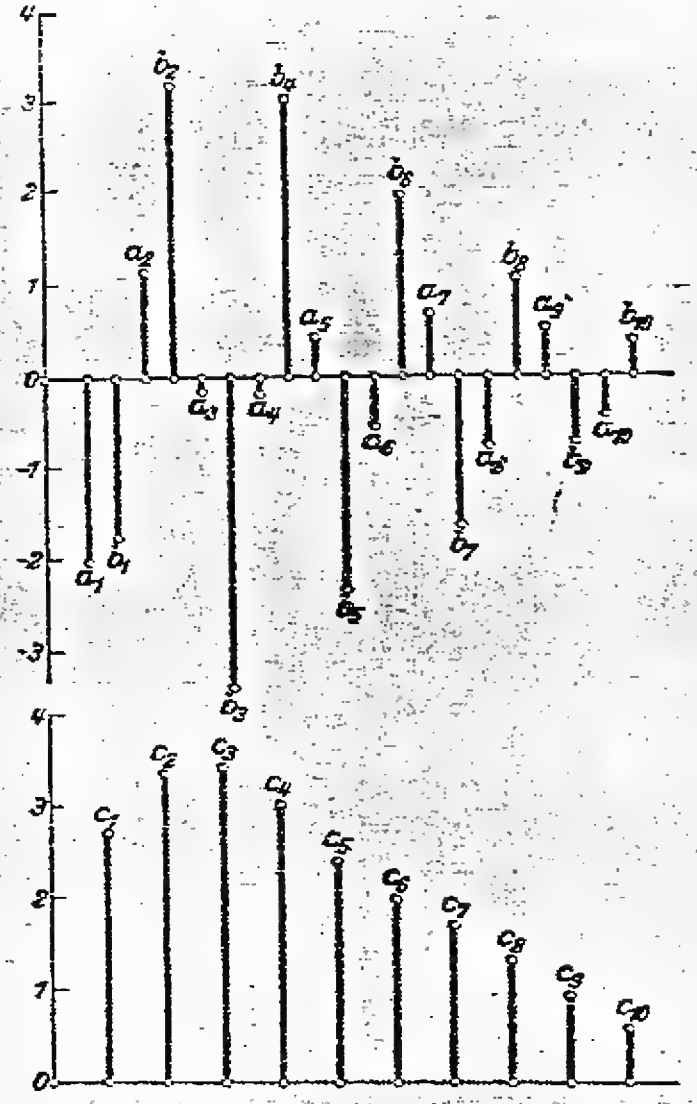


Fig. 6.—Fourier coefficients for the tangential force diagram of Fig. 4.

4. The Kinetic Energy.—In order to evaluate the next term of the power-balance equation (2, 2) or (2, 3) we designate the weights and dimensions of the crank mechanism by the same symbols G, G', G'', r, l, s', k and k' that were used in Sect. 2 of Part I. We remember that in Sect. 3 of that Part the connecting-rod was replaced by two weights, usually of finite dimensions: $G_1 = (1 - s'/l)G'$ with its centre of gravity on the axis of the crankpin A, and $G_2 = s'G'/l$, with its centre on the axis of the cross-head pin or gudgeon-pin B. Then, according to Fig. 1, the kinetic energy of the whole reciprocating mechanism is

$$T = \frac{1}{2g}\left\{\left[k^2 G + r^2\left(1 - \frac{s'}{l}\right)G'\right]\omega^2 + \left(\frac{s'}{l}G' + G''\right)v^2 + (k_1^2 G_1 + k_2^2 G_2)\dot{\chi}^2\right\}.$$

In this equation, k_1 and k_2 are the radii of gyration of the weights G_1 and G_2 with respect to the axes A and B, and are related by

$$k_1^2 G_1 + k_2^2 G_2 = [k^2 - s'(l - s')]G'.$$

We define the following quantities:

$$\left.\begin{aligned} R &= \frac{1}{g}\left[k^2 G + r^2\left(1 - \frac{s'}{l}\right)G'\right], \\ R' &= \frac{r^2}{g}\left(\frac{s'}{l}G' + G''\right), \\ R'' &= \frac{\lambda^2}{g}[s'(l - s') - k'^2]G' \quad \text{where } \lambda = \frac{r}{l}, \end{aligned}\right\} \tag{1}$$

and introduce them into the kinetic-energy relation, getting

$$T = \frac{1}{2}\Psi\omega^2 \quad \text{where} \quad \Psi = R + R'\left(\frac{r}{r\omega}\right)^2 - R''\left(\frac{\dot{\lambda}}{\lambda\omega}\right)^2. \tag{2}$$

The quantity Ψ, corresponding to the moment of inertia of a rotating rigid body, is called the effective moment of inertia of the reciprocating mechanism. Normally, the quantity R'' is much smaller numerically than R or R', since it contains the small factor λ^2 and since for connecting-rods the expression $[s'(l - s') - k'^2]$ is scarcely larger than r^2, if it does not actually vanish (Sect. 6 of Part I).

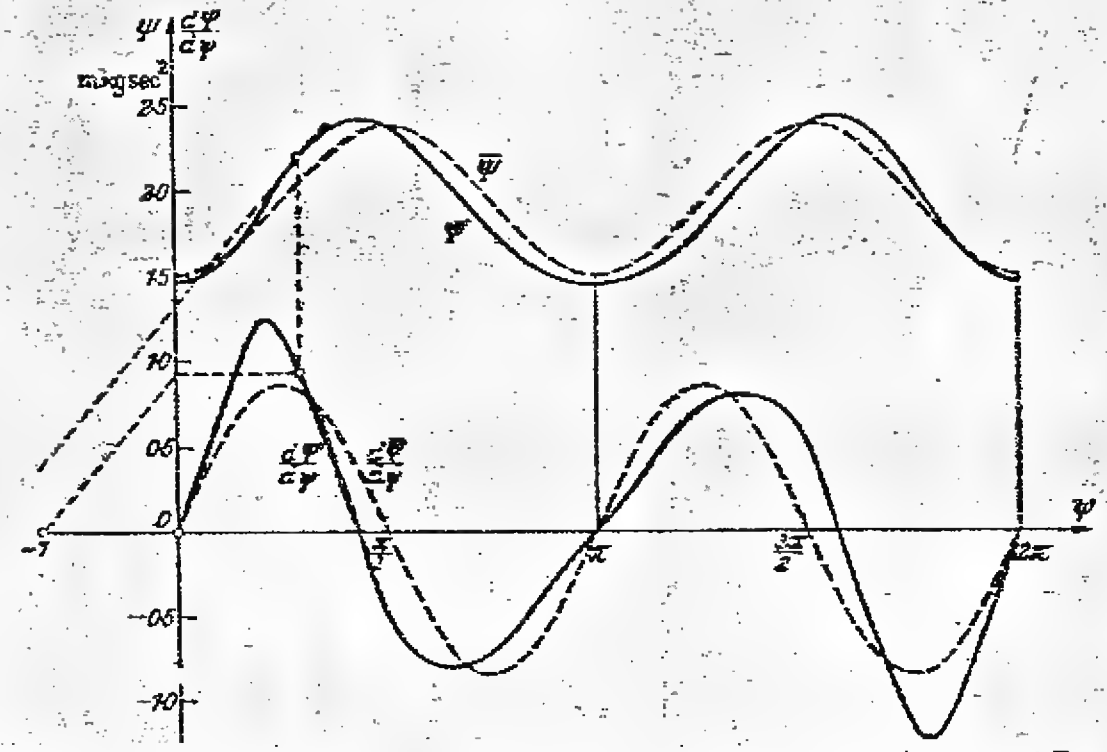


Fig. 7.—Effective moment of inertia Ψ of the standard reciprocating mechanism of Fig. 1, Part I, together with its approximation $\bar{\Psi}$ and the quantities $d\Psi/d\varphi$ and $d\bar{\Psi}/d\varphi$.

For finding the effective moment of inertia Ψ we shall develop a graphical and a numerical method. From Fig. 1,

$$\frac{v}{r\omega} = \frac{MB}{MA} = \frac{r'}{r}, \qquad \frac{\dot{\lambda}}{\lambda\omega} = \frac{l\dot{\alpha}}{r\omega} = \frac{l}{MA} = \frac{r''}{r},$$

from which

$$\Psi = R + R'\left(\frac{r'}{r}\right)^2 - R''\left(\frac{r''}{r}\right)^2. \tag{3}$$

Since r' and r'' vary with φ, the position of the crank, the quantity Ψ is also a function of φ. In order to find Ψ we must determine r' and r'' for various crank positions and then with the constants R, R' and R'' calculate (3). Fig. 7 shows the effective moment of inertia Ψ as a function of φ for the crank mechanism of the Diesel engine of Fig. 1 (Part I) with $\lambda = 0{\cdot}41$. In the standard mechanisms without offset the line of motion of the piston intersects the crank axis, and the curve for Ψ is always symmetric with respect to $\varphi = \pi$ and always repeats itself with period 2π.

For comparison, in Fig. 7 is shown (the broken line) the approximation due to Frahm, which is frequently used provided λ is not too large:

$$\overline{\Psi} = R + \tfrac{1}{2}R'(1 - \cos 2\psi). \tag{4}$$

This is obtained by assuming the angle χ (Fig. 1) to be negligibly small, so that $v = r\omega \sin\psi$, and by neglecting R'' in comparison to R. (According to (I, 4, 9) $\dot{\chi}/\lambda\omega$ varies between approximately $+1$ and -1.)

In order to express the effective moment of inertia by means of a formula involving the crank rotation ψ, we remember that the piston velocity v is the quantity represented by $-\dot{x}$ in Sect. 4 of Part I and can be expressed by means of (I, 4, 5) with the coefficients $B_j = A_j/j$ defined there:

$$\frac{v}{r\omega} = \sin\psi + B_2 \sin 2\psi - B_4 \sin 4\psi + B_6 \sin 6\psi - + \cdots.$$

Similarly, from (I, 4, 9) with $D_j = C_j/j$,

$$\frac{\dot{\chi}}{\lambda\omega} = D_1 \cos\psi - D_3 \cos 3\psi + D_5 \cos 5\psi - + \cdots.$$

Using the relations

$$2 \sin j\psi \sin k\psi = \cos(j-k)\psi - \cos(j+k)\psi,$$
$$2 \cos j\psi \cos k\psi = \cos(j-k)\psi + \cos(j+k)\psi,$$

we obtain the Fourier series

$$\left.\begin{aligned}\left(\frac{v}{r\omega}\right)^2 &= B'_0 + B'_1 \cos\psi - B'_2 \cos 2\psi - B'_3 \cos 3\psi - B'_4 \cos 4\psi - \\ &\quad + B'_5 \cos 5\psi + B'_6 \cos 6\psi - - + + \cdots, \\ \left(\frac{\dot{\chi}}{\lambda\omega}\right)^2 &= D'_0 + D'_2 \cos 2\psi - D'_4 \cos 4\psi + D'_6 \cos 6\psi - + \cdots,\end{aligned}\right\} \tag{5}$$

with coefficients

$$\left.\begin{aligned}
B'_0 &= \tfrac{1}{2}(1 + B_2^2 + B_4^2 + B_6^2 + \cdots), & B'_1 &= B_2, \\
B'_2 &= \tfrac{1}{2} + B_2 B_4 + B_4 B_6 + B_6 B_8 + \cdots, & B'_2 &= B_2 + B_4, \\
B'_4 &= \tfrac{1}{2} B_2^2 - B_2 B_6 - B_4 B_8 - B_6 B_{10} - \cdots, & B'_3 &= B_4 + B_6, \\
B'_6 &= B_2 B_4 - B_2 B_8 - B_4 B_{10} - B_6 B_{12} - \cdots, & B'_4 &= B_6 + B_8, \\
B'_8 &= \tfrac{1}{2} B_4^2 + B_2 B_6 - B_2 B_{10} - B_4 B_{12} - \cdots, & B'_5 &= B_8 + B_{10}, \\
B'_{10} &= B_2 B_8 + B_4 B_6 - B_2 B_{12} - B_4 B_{14} - \cdots, & B'_{11} &= B_{10} + B_{12}, \\
&\vdots & &\vdots \\
D'_0 &= \tfrac{1}{2}(D_1^2 + D_3^2 + D_5^2 + D_7^2 + \cdots), & D'_2 &= \tfrac{1}{2} D_1^2 - D_1 D_3 - D_3 D_5 - D_5 D_7 - \cdots, \\
D'_4 &= D_1 D_3 - D_1 D_5 - D_3 D_7 - D_5 D_9 - \cdots, & D'_6 &= \tfrac{1}{2} D_3^2 - D_3 D_5 - D_1 D_7 - D_3 D_9 - \cdots, \\
D'_8 &= D_1 D_7 + D_3 D_5 - D_1 D_9 - D_3 D_{11} - \cdots, & D'_{10} &= \tfrac{1}{2} D_5^2 + D_1 D_9 + D_3 D_7 - D_1 D_{11} - \cdots, \\
&\vdots & &\vdots
\end{aligned}\right\} \tag{6}$$

We know the laws governing the make-up of the coefficients, and for any

value of $\lambda = r/l$ we can quickly calculate the B_j' and D_j' from the B_j and D_j tabulated in (I, 4). Generally, it is permissible to use

$$\left.\begin{aligned} &B_0' = \tfrac{1}{2}(1 + B_2^2), \quad B_1' = B_2, \quad B_2' = \tfrac{1}{2}, \quad B_3' = B_2 + B_4, \quad B_4' = \tfrac{1}{2}B_2^2, \quad B_5' = B_4, \\ &D_0' = \tfrac{1}{2}D_2^2, \quad D_2' = \tfrac{1}{2}D_1^2 - D_1 D_3, \quad D_4' = D_1 D_3, \end{aligned}\right\} \tag{6a}$$

letting all other B_j' and D_j' be zero. In this way, using equations (2) and (5) we get the Fourier series for the kinetic energy T. The approximation (4) is obtained if, first, all B_j are put equal to zero, so that $B_0' = B_2' = \frac{1}{2}$ and all other $B_j' = 0$, and, second, if R'' is removed.

From (2, 3) the torque $\mathfrak{M}''$ corresponding to this kinetic energy is

$$\mathfrak{M}'' = \frac{dT}{d\psi} = \frac{1}{2}\left[\frac{d\Psi}{d\psi}\omega^2 + \Psi\frac{d(\omega^2)}{d\psi}\right] = \frac{1}{2}\frac{d\Psi}{d\psi}\omega^2 + \Psi\dot{\omega}. \tag{7}$$

Here, $d\Psi/d\psi$ is obtained either by graphical differentiation of the Ψ-curve as illustrated (for one point) in Fig. 7, or by analytical differentiation of the Fourier series obtained from (2) and (5). The approximation (4) gives

$$\overline{\mathfrak{M}}'' = \tfrac{1}{2}\omega^2 R' \sin 2\psi + \dot{\omega}[R + \tfrac{1}{2}R'(1 - \cos 2\psi)]. \tag{8}$$

In high-speed engines ω^2 is generally much larger than $\dot{\omega}$ (as will be shown in Sect. (7, 14)) so that in (7) or in (8) the second term ($\dot{\omega}$) can usually be neglected in comparison to the first (ω^2). The simplest approximation is, therefore,

$$\overline{\mathfrak{M}}'' = \tfrac{1}{2}\omega^2 R' \sin 2\psi. \tag{9}$$

5. **Potential Energy due to Gravity.**—In evaluating the potential-energy term of the power-balance equation (2, 2) or (2, 3) we must distinguish between mechanisms with horizontally and those with vertically moving pistons. Having consideration for future applications, we shall consider immediately the general case where the piston path makes an arbitrary angle δ with the vertical. With the notation of Fig. 8, where $x = OB$ and $s = OS$ represent the distances of point B and of the centre of gravity of the crank from the crank axis O, we express the potential energy of the mechanism for a given crank rotation ψ, taking as zero the energy when A is at its highest point A_0:

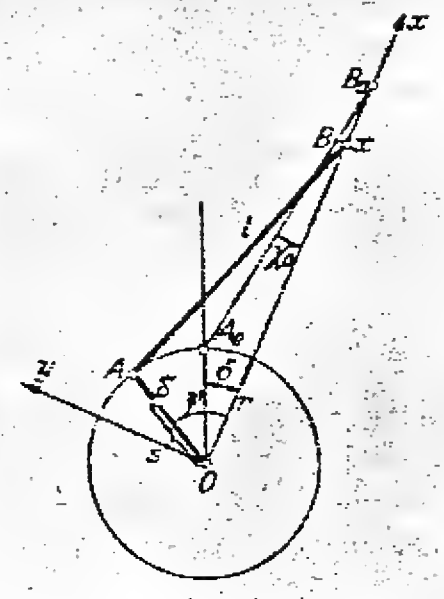


Fig. 8.—Standard reciprocating mechanism inclined from vertical.

$$U = -\left[sG + r\left(1 - \frac{s'}{l}\right)G'\right][1 - \cos(\psi - \delta)] - \left(\frac{s'}{l}G' + G''\right)(r\cos\delta + l\cos\chi_0 - x)\cos\delta.$$

Again, the weight of the connecting-rod is divided into two parts $(1 - s'/l)G'$ and $s'G'/l$ placed at the crankpin and at the cross-head or gudgeon-pin.

With the reduced first moments

$$Q^* = r\left(\frac{s'}{l}G' + G''\right), \qquad Q^{*\prime} = sG + r\left(1 - \frac{s'}{l}\right)G' \tag{1}$$

(defined somewhat differently than in I, 3) and with the values of x (I, 4, 5) we get the torque corresponding to the potential energy

$$\mathfrak{M}''' \equiv \frac{dU}{d\psi} = -\{(Q^* + Q^{*\prime})\sin\psi + Q^*(B_2\sin 2\psi - B_4\sin 4\psi + - \cdots)\}\cos\delta + \\ + Q^{*\prime}\cos\psi\sin\delta. \tag{2}$$

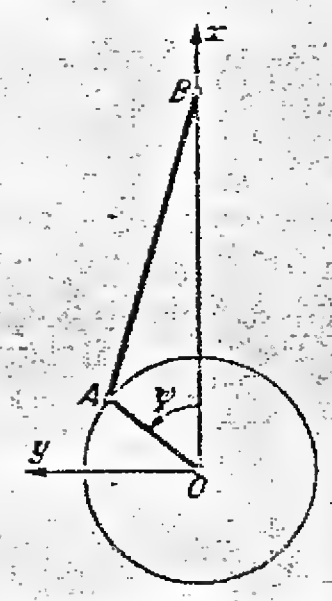


Fig. 9.—Upright engine.

Fig. 10.—Inverted engine.

On comparison with I, 5 the relationship with the inertia balancing of the engine immediately becomes clear. If X^* and Y^* designate the steady longitudinal and transverse forces we have

$$\mathfrak{M}''' = -\frac{g}{\omega^2}\left(\cos\delta\int_0^\psi X^*\,d\psi + \sin\delta\int_{\pi/2}^\psi Y^*\,d\psi\right). \tag{3}$$

Thus, the torque corresponding to the potential energy vanishes under the same conditions that make the longitudinal and transverse forces zero (I, 6). We note the following four especially important cases:

(a) The upright engine (Fig. 9). Since $\delta = 0$

$$\mathfrak{M}''' = -(Q^* + Q^{*\prime})\sin\psi - Q^*(B_2\sin 2\psi - B_4\sin 4\psi + - \cdots) = -\frac{g}{\omega^2}\int_0^\psi X^*\,d\psi. \tag{4}$$

(b) The inverted engine (Fig. 10). Since $\delta = \pi$

$$\mathfrak{M}''' = (Q^* + Q^{*\prime})\sin\psi + Q^*(B_2\sin 2\psi - B_4\sin 4\psi + - \cdots) = \frac{g}{\omega^2}\int_0^\psi X^*\,d\psi. \tag{5}$$

(c) The horizontal engine—Type I (Fig. 11). Since $\delta = \pi/2$

$$\mathfrak{M}''' = Q^{*\prime}\cos\psi = -\frac{g}{\omega^2}\int_{\pi/2}^\psi Y^*\,d\psi. \tag{6}$$

(d) The horizontal engine—Type II (Fig. 12). Since $\delta = -\pi/2$

$$\mathfrak{M}''' = -Q^{*\prime} \cos\psi = \frac{g}{\omega^2}\int_{\pi/2}^{\psi} \Gamma^* \, d\psi. \tag{7}$$

In cases (a) and (b) $\mathfrak{M}'''$ vanishes when the longitudinal force is balanced, and in cases (c) and (d) when the transverse force is balanced.

We observe that equation (3) is a special case of a much more general relationship which holds for all systems of one degree of freedom under the control of gravity. If m_i are the individual masses of the system and h_i the elevations of their centres of gravity above an arbitrary datum, the potential energy is

$$U = g \Sigma m_i h_i. \tag{8}$$

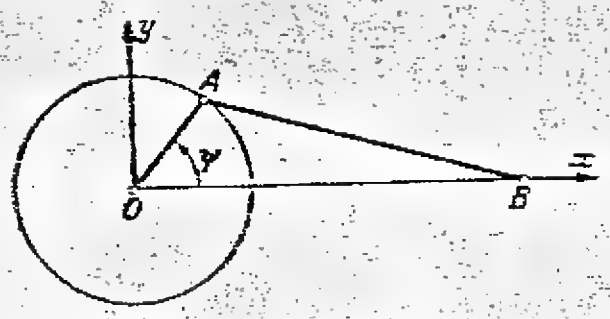


Fig. 11.—Horizontal engine—I.

Fig. 12.—Horizontal engine—II.

Moreover, if ψ is a coordinate that completely describes the configuration of the system, then $\mathfrak{M}'''$, the Lagrangian "force" obtained from the potential energy and corresponding to a virtual displacement $d\psi$, is given by

$$\mathfrak{M}''' \equiv \frac{dU}{d\psi}.$$

We define as "steady" any motion of the system in which $\omega = \dot{\psi}$ remains constant, so that from (8)

$$\frac{d\mathfrak{M}'''}{d\psi} \equiv \frac{d^2U}{d\psi^2} = g \Sigma m_i \frac{d^2 h_i}{d\psi^2} = \frac{g}{\omega^2} \Sigma m_i h_i.$$

But $-\Sigma m_i h_i$ is the vertical component K_y^* of the steady inertia force K^* (corresponding to constant ω), so that

$$\frac{d\mathfrak{M}'''}{d\psi} = -\frac{g}{\omega^2} K_y^*,$$

or

$$\mathfrak{M}''' = \mathfrak{M}_0''' - \frac{g}{\omega^2}\int_0^{\psi} K_y^* \, d\psi, \tag{9}$$

where $\mathfrak{M}_0'''$ is the value of $\mathfrak{M}'''$ at $\psi = 0$. This equation gives the desired general relation between the Lagrangian "force" $\mathfrak{M}'''$ corresponding to the

coordinate φ and due to the potential energy, and the steady inertia force K^* in a system with one degree of freedom.

In the previous case $K_h^* = X^* \cos\delta + Y^* \sin\delta$ (as indicated in Fig. 8) and $M_0''' = Q^{*\prime} \sin\delta = \sin\delta \frac{g}{\omega^2} \int_0^{\varphi=2\pi} Y^* d\varphi$ (I, 5, 2), so that (9) goes over into (3).

6. Internal Forces and Friction Losses.—The last term of the power balance equation contains the friction losses. These occur in the crank (main) bearings, crankpin bearings, cross-head or gudgeon-pin bearings, as well as between piston and cylinder walls (or cross-head and guides). In order to determine the friction losses we must first find the bearing forces and the wall pressure of the piston. Because this information is also important for other reasons (strength calculation of the connecting-rod, pins and bearings, for example) we shall obtain it in more detail and with greater accuracy than would be necessary only for determining friction losses.

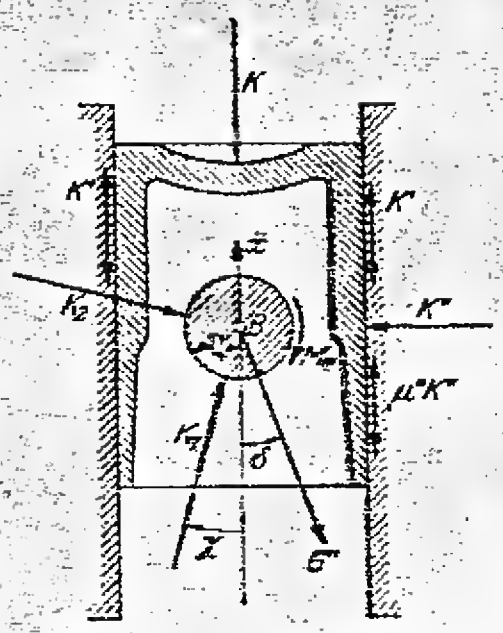


Fig. 13.—The piston and the forces on it.

Since we assume the piston force K to be known, as in Sect. 3, we start our computation at the piston end of the mechanism. Fig. 13 shows the piston with the forces acting on it: piston force K, unknown side-wall force K'', unknown force from the connecting-rod on the gudgeon-pin, which is resolved into a component K_1 along the rod and another, K_2, perpendicular to it, and the unknown moment M_z due to gudgeon-pin friction. (In Fig. 13, K and K'' are indicated as passing through point B. The actual resultant piston force and resultant side-wall force may be slightly offset; they can then be moved without rotation to B and supplemented by a couple. This couple need not be considered since it affects only the angular equilibrium of the non-rotating piston, and has no importance here.) There are, in addition, a friction force $\mu'' K''$ due to K'' (where μ'' is the friction coefficient) and a cylinder-wall friction force K' independent of K'' but dependent on the piston-ring pressure, the clearance between piston and cylinder wall, etc., and which must be assumed to be at least reasonably known for a given engine. The only other force on the piston is its weight G'', assumed to act at B, which must therefore be its centre of gravity. (This introduces no limitation; if the weight does not act exactly at B it can be moved there and combined with a couple, as was done with K and K''.) Since the following formulæ are to be applicable to engines working in any position, we will think of the weight G'' as acting at an angle δ with the direction of piston travel (shown vertical in Fig. 13). Hence, for the piston

with an acceleration $\ddot{x}$,

$$\left.\begin{aligned} K_1\cos\chi - K_2\sin\chi \pm \mu''|K''| &= K \mp K' + G''\left(\frac{\ddot{x}}{g}+\cos\delta\right),\\ K_1\sin\chi + K_2\cos\chi - K'' &= -G''\sin\delta. \end{aligned}\right\} \tag{1}$$

Here, the upper algebraic sign is used for the down-stroke ($0 \leq \psi \leq \pi$), the lower for the up-stroke ($\pi \leq \psi \leq 2\pi$).

We replace the connecting-rod by two equivalent masses G_2 and G_1 symmetric about the gudgeon-pin and the crankpin respectively (Fig. 14) which are connected by a weightless rigid member. On G_2, besides the reactions K_1, K_2 and M_1 from the gudgeon-pin, there act an unknown force from the rigid connection which is resolved into components K_3 and K_4 at the centre of gravity B and a couple M_2, as well as the weight $G_2 = s'G'/l$, at the angle δ with the cylinder axis. The corresponding reactions on G_1 from the rigid connection are K_3, K_4 acting at A and a couple M_3. Since the connection is weightless it follows that

$$lK_4 - M_2 - M_3 = 0. \tag{2}$$

On G_1, besides the crank force which we resolve into a tangential component P_1 and a normal component P_2, there act also the friction couple M_4 from the crankpin and the weight $G_1 = (1 - s'/l)G'$. Therefore, the following relations hold:

$$\left.\begin{aligned} (K_3 - K_1)\cos\chi + (K_2 - K_4)\sin\chi &= \frac{s'}{l}G'\left(\frac{\ddot{x}}{g}+\cos\delta\right),\\ (K_3 - K_1)\sin\chi - (K_2 - K_4)\cos\chi &= -\frac{s'}{l}G'\sin\delta. \end{aligned}\right\} \tag{3}$$

$$M_1 + M_2 = -G_2 l_2^2 \frac{\ddot{\chi}}{g}. \tag{4}$$

$$\left.\begin{aligned} P_1\sin\psi + P_2\cos\psi - K_3\cos\chi + K_4\sin\chi &= \left(1-\frac{s'}{l}\right)G'\left(\frac{\ddot{x}_1}{g}+\cos\delta\right),\\ -P_1\cos\psi + P_2\sin\psi + K_3\sin\chi + K_4\cos\chi &= \left(1-\frac{s'}{l}\right)G'\left(\frac{\ddot{y}_1}{g}+\sin\delta\right). \end{aligned}\right\} \tag{5}$$

$$M_3 + M_4 = -G_1 l_1^2 \frac{\ddot{\chi}}{g}. \tag{6}$$

Finally, on the crank (Fig. 15) in addition to the forces P_1, P_2 and M_4 from the connecting-rod, there are the main-bearing reactions P_3 and P_4, a couple $M_5 + \mathrm{L}$ whose first part M_5 is due to main-bearing friction and whose second part is the external load resistance (Sect. 8) and, lastly, the weight G. Then

$$\left.\begin{aligned} P_2 - P_4 &= G\left[\frac{s\omega^2}{g} - \cos(\psi - \delta)\right],\\ P_1 - P_3 &= G\left[\frac{s\dot{\omega}}{g} - \sin(\psi - \delta)\right], \end{aligned}\right\} \tag{7}$$

$$rP_1 - M_4 - M_5 = G\left[\frac{k^2\dot{\omega}}{g} - s\sin(\psi - \delta)\right] + \mathrm{L}. \tag{8}$$

To these equations we must add the friction laws: the relationships between the friction torques M_1, M_4, M_5 and the corresponding bearing reactions K_1 and K_2, P_1 and P_2, P_3 and P_4. Then it is possible to determine the unknowns K'', K_1, K_2, K_3, K_4, P_1, P_2, P_3, P_4, M_1, M_2, M_3, M_4, M_5 and, according to the problem, either the motion or the load resistance.

In the actual analysis we may permit ourselves to make many simplifications. The first consists in assuming the connecting-rod angle χ to be always small, and therefore,

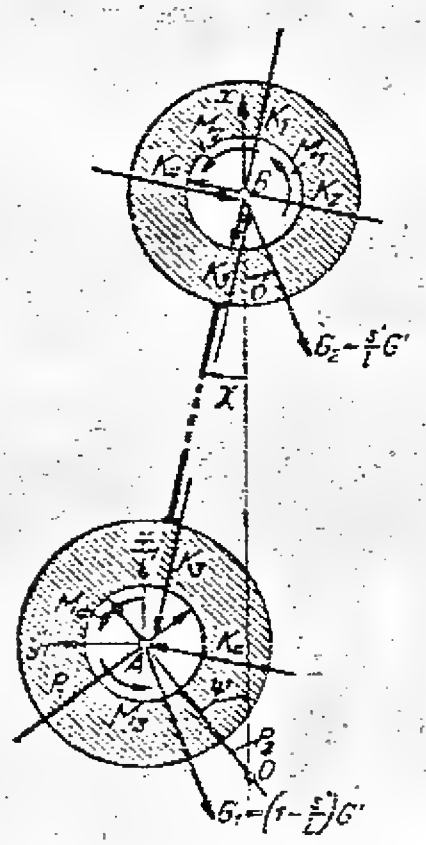


Fig. 14.—The mass-system representing the connecting-rod together with the forces on it.

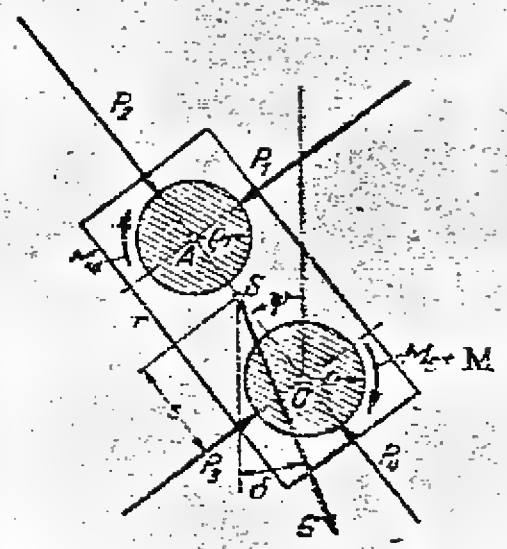


Fig. 15.—The crank and the forces on it.

likewise, the ratio $\lambda = r/l$. In most cases we put $\lambda = 0$ without hesitation; however, here we will be somewhat more careful and will use first-order terms in λ. Then, from (I, 4, 7) and (I, 4, 6),

$$\dot{x} = -r\omega(\sin\psi + \tfrac{1}{2}\lambda\sin 2\psi), \qquad \ddot{x} = -r\omega^2(\cos\psi + \lambda\cos 2\psi), \tag{9}$$

in which terms containing ω have been omitted. Likewise, from (I, 4, 1), (I, 4, 11) and (I, 4, 10)

$$\sin\chi = \lambda\sin\psi, \quad \cos\chi = 1, \quad \dot{\chi} = \lambda\omega\cos\psi, \quad \ddot{\chi} = -\lambda\omega^2\sin\psi. \tag{10}$$

The components of the acceleration $\ddot{x}_1$, $\ddot{y}_1$ of the crankpin A are

$$\ddot{x}_1 = -r\omega^2\cos\psi, \qquad \ddot{y}_1 = -r\omega^2\sin\psi. \tag{11}$$

Moreover, in the first of equations (1) the friction force $\mu''K''$ remains very small because of the smallness of the friction coefficient μ'' (generally less than 1/10) and may be ignored in comparison with the piston force. Likewise, K' can be ignored here. (This does not mean that later we will neglect

the friction losses due to these forces, but only that we consider their effect on the bearing forces to be unimportant.) By addition of equations (2), (4) and (6) we obtain

$$lK_4 + M_1 + M_4 = -(k_1^2 G_1 + k_2^2 G_2)\frac{\ddot{\gamma}}{g}. \tag{12}$$

Combining this with (10) and with (I, 8, 5) gives the crankpin force

$$K_4 = -\frac{M_1 + M_4}{l} - \frac{s'(l-s') - k'^2}{l^2}\frac{r\omega^2}{g} G' \sin\psi. \tag{13}$$

It is apparent that this force is always very small since the friction torques M_1 and M_4, with a bearing radius ρ and friction coefficient μ, are of the order of magnitude $\mu\rho K$, while the ratio $[s'(l-s') - k'^2] : l^2$ is always a very small fraction if not zero (I, 6). Therefore, we simplify the calculation greatly by assuming that

$$K_4 = 0. \tag{14}$$

Next, we introduce equations (9) to (11) into (1), (3) and (5).

Then, from (1) and (3) we get immediately the gudgeon-pin reactions

$$\left.\begin{aligned} K_1 &= K - G''\left[\frac{r\omega^2}{g}(\cos\psi - \lambda\cos 2\psi) - \cos\delta\right] + \frac{s'}{l} G' \lambda \sin\delta \sin\psi, \\ K_2 &= \frac{s'}{l} G'\left[\sin\delta - \lambda\left(\frac{1}{2}\frac{r\omega^2}{g}\sin 2\psi - \cos\delta\sin\psi\right)\right], \end{aligned}\right\} \tag{15}$$

the side-wall force

$$K'' = \lambda K \sin\psi + \left(\frac{s'}{l} G' + G''\right)\left[\sin\delta - \lambda\left(\frac{1}{2}\frac{r\omega^2}{g}\sin 2\psi - \cos\delta\sin\psi\right)\right] \tag{16}$$

and the connecting-rod force

$$K_s = K - \left(\frac{s'}{l} G' + G''\right)\left[\frac{r\omega^2}{g}(\cos\psi + \lambda\cos 2\psi) - \cos\delta\right]. \tag{17}$$

In the analysis we have always neglected terms containing λ^2. The connecting-rod force K_2 is not real, but is an imaginary force in the assumed weightless connection between the equivalent masses G_1 and G_2 which replace the connecting-rod. The actual force that acts in the connecting-rod is, at any instant, variable along the rod (on account of its inertia), being given at end B by the components K_1, K_2 and at A by P_1, P_2. However, we may assume K_s to be the mean value of the longitudinal connecting-rod force, and without hesitation use it in the strength analysis of the rod.

With the help of (14) and (17), equations (5) finally give the two components of the crankpin reaction,

$$\left.\begin{aligned} P_1 &= K\left(\sin\psi+\frac{1}{2}\lambda\sin 2\psi\right)+\left(1-\frac{s'}{l}\right)G'\sin(\psi-\delta)- \\ &\quad -\left(\frac{s'}{l}G'+G''\right)\left\{\frac{1}{2}\frac{r\omega^2}{g}\left[\sin 2\psi-\frac{1}{2}\lambda(\sin\psi-3\sin 3\psi)\right]- \right. \\ &\quad \left. -\cos\delta\left(\sin\psi+\frac{1}{2}\lambda\sin 2\psi\right)\right\}, \\ P_2 &= K\left[\cos\psi-\frac{1}{2}\lambda(1-\cos 2\psi)\right]-\left(1-\frac{s'}{l}\right)G'\left[\frac{r\omega^2}{g}-\cos(\psi-\delta)\right]- \\ &\quad -\left(\frac{s'}{l}G'+G''\right)\left\{\frac{1}{2}\frac{r\omega^2}{g}\left[1+\cos 2\psi+\frac{1}{2}\lambda(\cos\psi+3\cos 3\psi)\right]- \right. \\ &\quad \left. -\cos\delta\left[\cos\psi-\frac{1}{2}\lambda(1-\cos 2\psi)\right]\right\}. \end{aligned}\right\} \tag{18}$$

Since the piston force K is represented graphically as a function of ψ (Fig. 2 of Sect. 3) it is best to evaluate the first terms of equation (18) graphically also. From the triangles OAD and OAI (where OI is perpendicular to AD) of Fig. 2 of Sect. 3,

$$OD = r\frac{\sin(\psi+\chi)}{\cos\chi}, \qquad AI = r\cos(\psi+\chi).$$

Combining with (10) gives

$$OD = r(\sin\psi+\tfrac{1}{2}\lambda\sin 2\psi), \qquad AI = r[\cos\psi-\tfrac{1}{2}\lambda(1-\cos 2\psi)].$$

It follows that

$$\left.\begin{aligned} K\left(\sin\psi+\frac{1}{2}\lambda\sin 2\psi\right) &= K\frac{OD}{r} = OF \equiv P, \\ K\left[\cos\psi-\frac{1}{2}\lambda(1-\cos 2\psi)\right] &= K\frac{AI}{r} = EL \end{aligned}\right\} \tag{18a}$$

These expressions are suitable for graphical analysis; the first simply gives the tangential force P (Sect. 3); the second is equally easy to find.

Lastly, from equation (7) with $\dot\omega = 0$, we get the two components of the main bearing force:

$$\left.\begin{aligned} P_3 &= P_1 + G\sin(\psi-\delta), \\ P_4 &= P_2 - G\left[\frac{r\omega^2}{g}-\cos(\psi-\delta)\right], \end{aligned}\right\} \tag{19}$$

in which the values of (18) are to be inserted. (For the time being, we have no need of equation (8) itself.)

Thus all inter-member forces are determined. The most important terms in equations (15)–(19) are naturally those containing the piston force K. In high-speed engines, the inertia terms with the factor $r\omega^2/g$ are roughly of the same order and may not be neglected, even in an approximate analysis. The terms arising from the gravity forces, recognizable through the factor $\cos\delta$ or $\sin\delta$, as well as $\cos(\psi-\delta)$ or $\sin(\psi-\delta)$, become relatively un-

important at high speeds. For a first approximation we may omit terms containing λ.

We now proceed to calculate the power loss V due to friction, and the torque $M'''' \equiv V/\omega$ that corresponds to it. The first loss V_1 is due to the side-wall friction force K'. It equals $K' |\dot{x}|$. Taking $\dot{x}$ from (9) and dividing by ω, we get

$$M_1'''' = rK' |\sin\varphi + \tfrac{1}{2}\lambda \sin 2\varphi|. \tag{20}$$

The side-wall normal force K'' (16) determines the second part of the lost torque $M_2'''' = \mu'' |K''| \cdot |\dot{x}|/\omega$, or, in full,

$$M_2'''' = \mu'' r \left| \frac{1}{2}\lambda K (1 - \cos 2\varphi) + \left(\frac{\delta'}{l} G' + G''\right) \left\{ \sin\delta \sin\varphi - \right. \right.$$
$$\left. \left. - \frac{1}{2}\lambda \left[\frac{1}{2} \frac{r\omega^2}{g} (\cos\varphi - \cos 3\varphi) - \cos\delta (1 - \cos 2\varphi) - \sin\delta \sin 2\varphi \right] \right\} \right|. \tag{21}$$

At the gudgeon-pin there is the reaction $Z_2 = +\sqrt{(K_1^2 + K_2^2)}$; from the friction coefficient μ_2' and the radius ρ_2 of the gudgeon-pin, as well as its angular velocity $\dot{\chi}$ (10), the power loss at the gudgeon-pin is $V_2 = \mu_2' \rho_2 Z_2 |\dot{\chi}|$, and the corresponding torque is

$$M_3'''' = \mu_2' \rho_2 \lambda Z_2 |\cos\varphi|. \tag{22}$$

Similarly, from the crankpin reaction $Z_1 = +\sqrt{(P_1^2 + P_2^2)}$, the friction coefficient μ_1', the crankpin radius ρ_1 and the relative angular velocity between crank and connecting-rod, we get the torque loss

$$M_4'''' = \mu_1' \rho_1 Z_1 (1 + \lambda \cos\varphi). \tag{23}$$

Finally, the main-bearing reaction $Z_3 = +\sqrt{(P_3^2 + P_4^2)}$, the bearing-friction coefficient μ and the main-bearing radius ρ give the corresponding torque loss

$$M_5'''' = \mu \rho Z_3. \tag{24}$$

The bearing-friction factors μ, μ_1', μ_2', as well as the coefficient μ'', are assumed to be fairly well known [1] with respect to dependence not only on velocity but also on pressure and temperature, so that no uncertainties arise in the calculation of the total friction torque

$$M'''' \equiv \frac{V}{\omega} = M_1'''' + M_2'''' + M_3'''' + M_4'''' + M_5''''. \tag{25}$$

The actual calculation for a sufficient number of crank positions φ is very time-consuming, particularly on account of the complicated expressions for Z_1, Z_2 and Z_3. Usually, we simplify this task considerably, bearing in mind that the uncertainties in the friction coefficients and the smallness of the whole friction torque make a very accurate calculation unprofitable,

[1] See, for example, "Hütte", Vol. 1, p. 286; Vol. 2, p. 110, 25th ed., Berlin 1925; also M. Ewriss, "Reibungs- und Undichtigkeitsverluste an Kolbenringen", *Forsch.-Arb. Ing. Wes.*, Vol. 371 (1935).

by omitting all terms that depend only on weight, as well as those containing λ, so long as they are not multiplied by the normally large piston force K or the factor $r\omega^2/g$. Moreover, terms containing the product $\rho_1\lambda$ or $\rho_2\lambda$ can be dropped.

With these approximations we have

$$\mathfrak{M}_1'''' = rK'|\sin\varphi|, \qquad \mathfrak{M}_2'''' = \mu''r\lambda B_1\sin^2\varphi, \qquad \mathfrak{M}_3'''' = 0. \tag{26}$$

Also

$$\left.\begin{aligned} Z_1^2 &= H_1^2 + H_2^2 - 2H_1H_2\cos\varphi, \\ Z_2^2 &= H_1^2 + H_3^2 - 2H_1H_3\cos\varphi, \end{aligned}\right\} \tag{27}$$

where

$$\left.\begin{aligned} B_1 &= K - Q_1\omega^2\cos\varphi, & Q_1 &= \frac{r}{g}\left(\frac{\varepsilon'}{l}G' + G''\right), \\ H_2 &= Q_2\omega^2, & Q_2 &= \frac{r}{g}\left(1 - \frac{\varepsilon'}{l}\right)G', \\ H_3 &= Q_3\omega^2, & Q_3 &= \frac{1}{g}\left[\varepsilon G + r\left(1 - \frac{\varepsilon'}{l}\right)G'\right]. \end{aligned}\right\} \tag{28}$$

and finally

$$\mathfrak{M}_1'''' = \mu_1'\varrho_1 Z_1, \qquad \mathfrak{M}_2'''' = \mu_2\varrho_2 Z_2. \tag{29}$$

Fig. 16 shows $\mathfrak{M}_1''''$, the torque loss due to piston friction K'. Fig. 17 illustrates the graphical determination of the torque loss $\mathfrak{M}_2''''$ and the crank force Z_1 from the piston force K and the inertia forces $Q_1\omega^2$ and $H_2 = Q_2\omega^2$. On the diameter of the crank circle (defined by upper and lower dead-centres) we draw two semicircles with radii $Q_1\omega^2$ and $Q_2\omega^2$, placing the indicator diagram for piston force K above. In accordance with the present accuracy (neglecting terms with λ), K will be found by projecting directly upward from A. The value of $Q_1\omega^2\cos\varphi$ corresponding to the crank rotation φ is subtracted from K (taking account of algebraic signs). In this way H_1 is found,

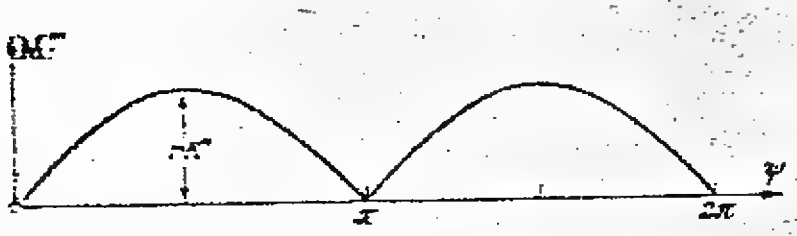


Fig. 16.—Torque loss due to piston friction.

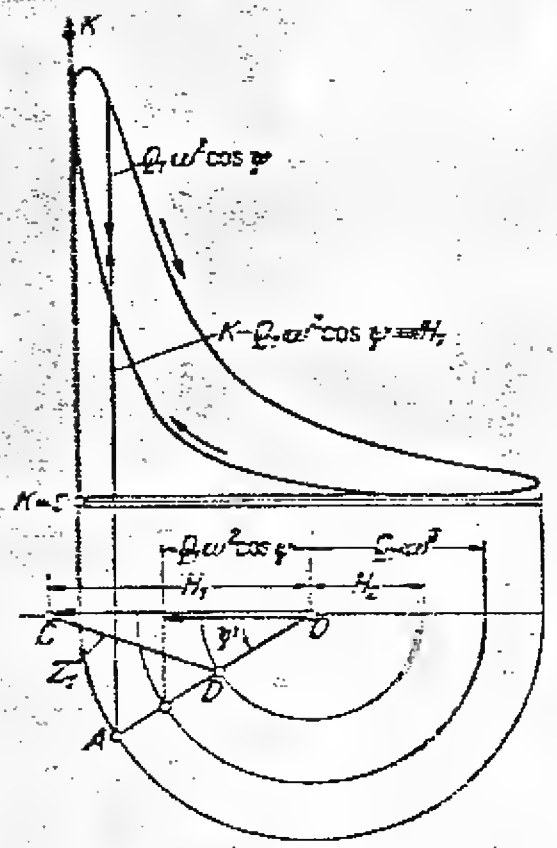


Fig. 17.—Graphical determination of the torque loss $\mathfrak{M}_2''''$ and the crankpin reaction Z_1.

from which the torque $\mathfrak{M}_2''''$ follows through multiplication by $\mu''r\lambda\sin^2\varphi$. Next, the difference H_1 (again taking account of algebraic signs) is laid off

from O to C, positive to the left. From equation (27) the (always positive) distance CD shows the crankpin force Z_1 in both magnitude and direction. The same construction, with $H_3 = Q_3 r\omega^2$ instead of $H_2 = Q_2 r\omega^2$, yields the main-bearing reaction Z_3. For the interval $\pi < \psi < 2\pi$ we continue to use the same crank semicircle with angles reflected in its diameter. When $K = 0$, we need only lay off the interval $Q_1 r\omega^2 \cos\psi$ along the diameter. If $s = 0$ through use of counterweights, then $Q_2 = Q_3$ and the bearing reactions Z_1 and Z_3 are equal. If $Q_3 = 0$, possibly because of inertia balancing (see I, 6, where Q' is used instead of Q_3), the construction for Z_3 is particularly simple since $Z_3 = |H_1|$.

Fig. 18 shows, for illustration, the crankpin reaction Z_1 as a function of ψ for the four-stroke Diesel engine illustrated in Fig. 1 of Part I, for 300 and for 600 r.p.m. The diagram for the main-bearing reaction would be

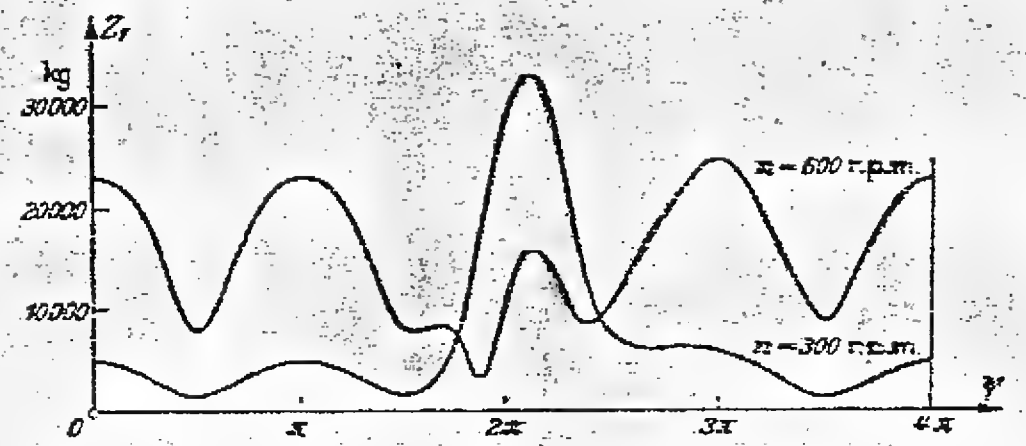


Fig. 18.—The crankpin reaction for the mechanism of Fig. 1, Part I, with the indicator diagram of Fig. 17 for small and large r.p.m.

similar. At low speeds, the crankpin force Z_1 follows the piston force approximately, having a maximum at the instant of firing. At higher speeds, it appears that the reaction during the working stroke is appreciably smaller than during the non-working strokes; the inertia of the rapidly accelerated masses causes unloading of the bearings during the working stroke. The centripetal acceleration of the equivalent mass imagined to be at the crankpin cancels the greatest part of the piston force; thus in high-speed engines the calculated tangential reaction P is not even approximately equal to the actual force.

The torque losses (29) are finally obtained from the bearing reactions multiplied by the so-called friction radii $\mu' r_1$ and $\mu\rho$.

We can now easily discover whether neglecting the gravity forces is permissible or not, by inserting these terms alone into equations (15)–(19). In this way, we find that with the same degree of approximation used in (26), for the upright engine ($\delta = 0$)

$$\bar{F}'' = \lambda\left(\frac{s'}{l}G' + G''\right)\sin\psi, \qquad \bar{Z}_1 = G' + G'', \qquad \bar{Z}_3 = G + G' + G'', \tag{30}$$

and for the horizontal engine ($\delta = \pi/2$)

$$\bar{K}'' = \frac{\delta'}{l} G' + G'', \qquad \bar{Z}_1 = \left(1 - \frac{\delta'}{l}\right) G', \quad \bar{Z}_3 = G + \left(1 - \frac{\delta'}{l}\right) G'. \tag{31}$$

From these, for comparison with (26) and (29), we calculate the friction torques due to gravity forces alone:

$$\bar{\mathfrak{M}}_2'''' = \mu'' r \left| \bar{K}'' \sin \psi \right|, \qquad \bar{\mathfrak{M}}_4'''' = \mu_1' \varrho_1 \bar{Z}_1, \qquad \bar{\mathfrak{M}}_5'''' = \mu_0 \varrho \bar{Z}_3. \tag{32}$$

On account of the non-linear relationship between the friction torques and the components of the bearing reactions it is not permissible to add the torques $\bar{\mathfrak{M}}_4''''$ and $\bar{\mathfrak{M}}_5''''$ directly to (29). Instead, when it appears that the torques (32) may not be neglected, the bearing reactions Z_1 and Z_3 must be recalculated, and from them $\mathfrak{M}_4''''$ and $\mathfrak{M}_5''''$. On the other hand, $\bar{\mathfrak{M}}_2''''$ may be added to $\mathfrak{M}_2''''$ (26) directly.

7. **Coefficient of Cyclic Variation and Angular Deviation.**—At last, all terms of the power-balance equation (2, 2) or (2, 3) are known as functions of ψ, $\omega = \dot{\psi}$ and $\dot{\omega} = \ddot{\psi}$, and can be used to determine the useful torque $\mathfrak{M}^{(n)}$ if ω and $\dot{\omega}$ are given as functions of the crank rotation ψ or of time. In practice, the problem is always so stated that the average angular velocity ω_m, from which the number of revolutions per second or per minute is easily found, is specified, and certain additional specifications regarding the uniformity of the motion are included. In general, it is required that the variation of the angular velocity ω remain within certain limits, which may be fixed by the requirements of the apparatus being driven (pump, generator, etc.). Occasionally, instead of this there may be specified a limit for the deviation of the angle of rotation from its ideal value $\psi_m = \omega_m t$, as in parallel driving of alternators. These supplementary requirements can generally be fulfilled only if the engine has a means of storing energy in the form of a flywheel. Our task is to determine the required moment of inertia A of the flywheel from the given specifications. Sometimes the problem is to determine the degree of non-uniformity of rotation or the angular deviation in an engine without a flywheel or with a moment of inertia fixed by other considerations (e.g. a propeller).

We specify a stationary state, i.e. we assume that conditions are repeated after every revolution with two-stroke engines, or after every two revolutions with four-stroke engines. Since $\omega = d\psi/dt$ the length of such a period is

$$t_0 = \int_0^{4\pi} \frac{d\psi}{\omega}, \tag{1}$$

for the more important case of the four-stroke engine. With the two-stroke engine the integration interval is naturally 2π. Since $t_0 = 2/n$ for four-

stroke engines, $\omega_m = 4\pi/t_0$; the average angular velocity ω_m is defined by the relation

$$\frac{1}{\omega_m} = \frac{1}{4\pi}\int_0^{4\pi} \frac{d\psi}{\omega} \tag{2}$$

This average value is important not only in connection with the revolutions per minute or per second, but also for the average power N_m of the engine. Let us define

$$\mathbf{M}_m^{(n)} = \frac{1}{4\pi}\int_0^{4\pi} \mathbf{M}^{(n)}\, d\psi \tag{3}$$

as the average useful torque of the engine (i.e. as the constant torque that does the same work per period as the actual torque $\mathbf{M}^{(n)}$); with $\omega_m = 4\pi/t_0$ it follows that

$$\mathbf{M}_m^{(n)}\omega_m = \frac{1}{t_0}\int_0^{4\pi} \mathbf{M}^{(n)}\, d\psi = \frac{1}{t_0}\int_0^{t_0} \mathbf{M}^{(n)}\omega\, dt = \frac{1}{t_0}\int_0^{t_0} N\, dt.$$

From this, the average power, usually simply called the power, can be calculated:

$$N_m = \frac{1}{t_0}\int_0^{t_0} N\, dt, \tag{4}$$

It is seen to be simply the product of $\mathbf{M}_m^{(n)}$ and ω_m,

$$N_m = \mathbf{M}_m^{(n)}\omega_m, \tag{5}$$

provided we use the previous definition (2) of average angular velocity ω_m. Then the ideal crank rotation for uniform motion is

$$\psi_m = \omega_m t. \tag{6}$$

Besides the average angular velocity ω_m we must consider the arithmetic mean $\bar{\omega}$ between $\omega_{\max}$ and $\omega_{\min}$ for each interval:

$$\bar{\omega} = \tfrac{1}{2}(\omega_{\max} + \omega_{\min}). \tag{7}$$

Since in a stationary state ω is a periodic function of ψ,

$$\omega = \bar{\omega}[1 + \tfrac{1}{2}\varepsilon p(\psi)], \tag{8}$$

where ε is a factor, normally small, which we may take as positive, and $p(\psi)$ is a periodic function of ψ (with period 2π or 4π in two-stroke or four-stroke engines) which we can specify to alternate between $+1$ and -1 so that

$$\omega_{\max} = \bar{\omega}(1 + \tfrac{1}{2}\varepsilon), \qquad \omega_{\min} = \bar{\omega}(1 - \tfrac{1}{2}\varepsilon). \tag{9}$$

The factor ε is called the coefficient of cyclic variation and describes the degree of irregularity of an engine. It is defined by

$$\varepsilon = \frac{\omega_{\max} - \omega_{\min}}{\bar{\omega}}. \tag{10}$$

If a particular level of uniformity of operation is specified, this means that ε has an upper limit which must not be exceeded, for example, 1/40 with machine tools, 1/300 with alternators. Practically, the coefficient of cyclic variation is usually defined by the expression

$$\varepsilon' = \frac{\omega_{max} - \omega_{min}}{\omega_m} \tag{11}$$

instead of by (10). It will now be shown that for small values of ε and ε' the two definitions (10) and (11) are practically indistinguishable numerically.

From (2) and (8) it follows for $\varepsilon < 2$ that

$$\frac{\bar{\omega}}{\omega_m} = \frac{1}{4\pi}\int_0^{4\pi} \frac{d\psi}{1 + \frac{1}{2}\varepsilon p(\psi)} = \frac{1}{4\pi}\int_0^{4\pi} \left[1 - \frac{1}{2}\varepsilon p(\psi) + \frac{1}{4}\varepsilon^2 p^2(\psi) - + \cdots\right] d\psi,$$

or, approximately, for small values of ε

$$\frac{\bar{\omega}}{\omega_m} = 1 - \frac{1}{2}\varepsilon p_m \quad \text{with} \quad p_m = \frac{1}{4\pi}\int_0^{4\pi} p(\psi)\, d\psi. \tag{12}$$

The average p_m of the function $p(\psi)$ is not necessarily zero, but it is always rather small since the angular velocity ω and therefore the periodic function itself oscillate rather uniformly between their limits (Fig. 21 of Sect. 9). It follows that

$$\frac{\omega_m - \bar{\omega}}{\omega_m} \ll \frac{1}{2}\varepsilon, \tag{13}$$

so that practically we may always interchange $\bar{\omega}$ and ω_m as well as ε and ε' without concern.

For the dimensionless ratio $\dot{\omega}/\omega^2$ we get from (8)

$$\frac{\dot{\omega}}{\omega^2} = \frac{1}{\omega}\frac{d\omega}{d\psi} = \frac{\frac{1}{2}\varepsilon \frac{dp}{d\psi}}{1 + \frac{1}{2}\varepsilon p}.$$

If ε is small this is very nearly

$$\frac{\dot{\omega}}{\omega^2} = \frac{1}{2}\varepsilon \frac{dp}{d\psi}\left(1 - \frac{1}{2}\varepsilon p\right) \sim \frac{1}{2}\varepsilon, \tag{14}$$

because the derivative $dp/d\psi$ is at most of the order of p itself, and therefore can hardly exceed the limits ± 1. Thus, it is shown that very often in approximate calculations the terms in $\dot{\omega}$ can be neglected by comparison with terms in ω^2 (Sect. 4 and 6 and the last part of I, 5).

For later use we note that the usual average value of ω^2,

$$(\omega^2)_m = \frac{1}{4\pi}\int_0^{4\pi} \omega^2\, d\psi, \tag{15}$$

can be obtained from (8) and (12) by dropping terms of the order ε^2, in the form

$$(\omega^2)_m = \frac{\bar{\omega}^2}{4\pi}\int_0^{4\pi}(1+\varepsilon\eta)\,d\psi = \omega_m(1-\varepsilon\eta_m)(1+\varepsilon\eta_m)$$

or

$$(\omega^2)_m = \omega_m^2. \tag{16}$$

so that for small ε the quantity $(\omega^2)_m$ can be immediately interchanged with ω_m^2.

The angular deviation is defined by the expression

$$\Delta\psi = \psi - \psi_m \quad \text{with} \quad \psi_m = \omega_m t, \tag{17}$$

which is related very simply to the quantity

$$\zeta = \frac{\omega - \omega_m}{\omega_m}, \tag{18}$$

which we may call the speed deviation. From $\omega = d\psi/dt$ follows

$$\psi_m = \omega_m t = \omega_m\int_0^{\psi}\frac{d\psi}{\omega} = \int_0^{\psi}\frac{d\psi}{1+\zeta}.$$

Since in any case $\zeta < \varepsilon'$ we can neglect higher powers of ζ with at least the same justification as in neglecting higher powers of ε, so that

$$\psi_m = \int_0^{\psi}(1-\zeta)\,d\psi = \psi - \int_0^{\psi}\zeta\,d\psi.$$

Therefore

$$\Delta\psi = \int_0^{\psi}\zeta\,d\psi. \tag{19}$$

This shows how the angular deviation $\Delta\psi$ is obtained very exactly from the speed deviation ζ.

8. Flywheel Calculation; First Method.—We now return to the power balance of the engine and in view of the unavoidable inaccuracy of the indicator diagram and the friction coefficients we conclude that no loss in accuracy will result if the variable factor ω^2 in the torques $\mathfrak{M}''$ and $\mathfrak{M}''''$ (Sect. 4 and 6) is replaced by its average value ω_m^2, and, for the time being, the term $\dot{\omega}$ in $\mathfrak{M}''$ is ignored (as was done in $\mathfrak{M}''''$). Then, with abscissa ψ, we plot the curve for the tangential-force torque $\mathfrak{M}'$ (Sect. 3) as well as the curves for $\mathfrak{M}''$, $\mathfrak{M}'''$ and $\mathfrak{M}''''$ (Sect. 4 to 6) with $\omega^2 = \omega_m^2$ and $\dot{\omega} = 0$, obtaining by subtraction the function

$$\mathfrak{M}^{(a)} = \mathfrak{M}' - (\mathfrak{M}'' + \mathfrak{M}''' + \mathfrak{M}''''). \tag{1}$$

Fig. 19 shows this function for an upright four-stroke Diesel engine having the standard reciprocating mechanism of Fig. 1 (Part I), the indicator diagram of Fig. 2, the tangential-force torque of Fig. 4 of Sect. 3, the effective moment of inertia of Fig. 7, Sect. 4, and the bearing forces of Fig. 18, Sect. 6. The speed is 600 r.p.m., the friction radii of the bearings are $\mu'_1\rho_1 = \mu\rho = 0{\cdot}005$ m. and the friction coefficient at the cylinder wall is such that $\mu''r = 0{\cdot}02$ m.

The ordinates of the curve show the useful torque $\mathbf{M}^{(n)}$ that must be taken from the start if the motion is to be steady and uniform. The actual torque $\mathbf{L}$ taken from the crank, the so-called load resistance, will normally obey quite a different law. (We need hardly mention that $\mathbf{L}$, both here and later,

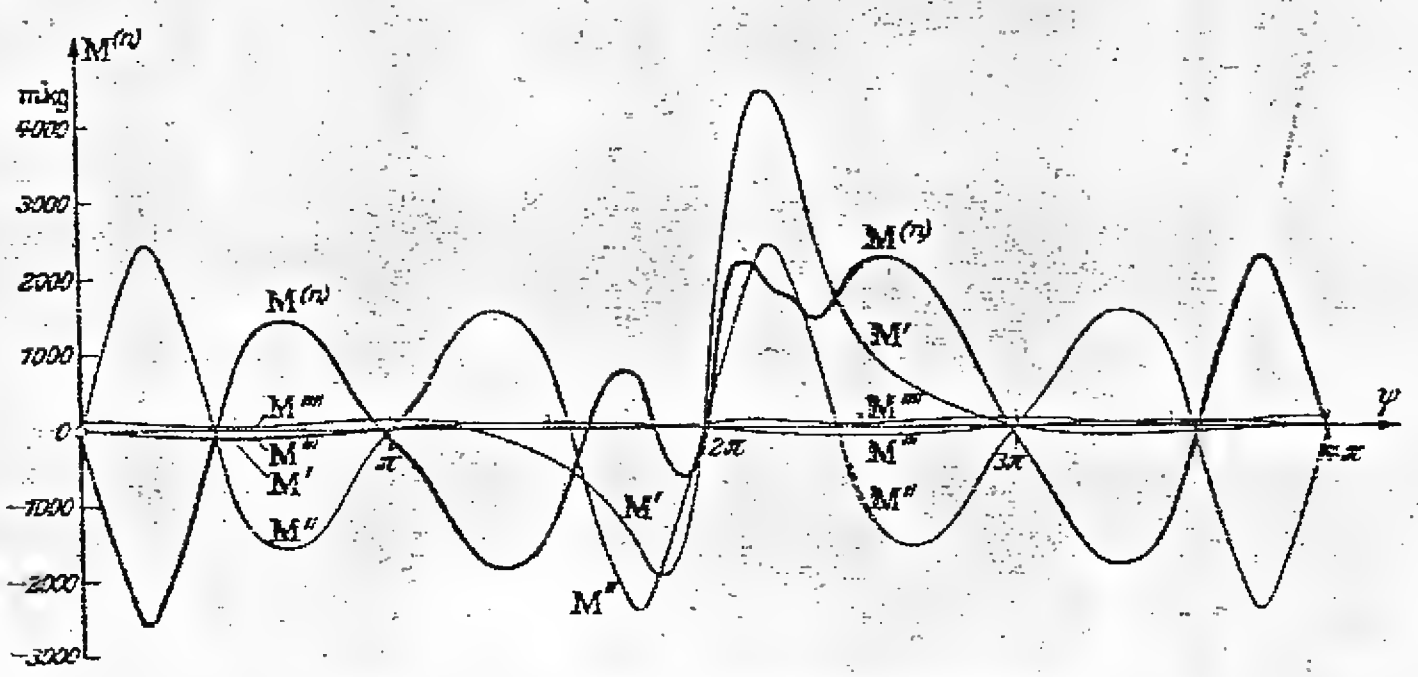


Fig. 19.—The useful torque of a four-cycle Diesel engine.

represents not only the mechanical-energy output but also the losses due to the various pieces of supplementary apparatus driven by the crank, such as fan, injector, ignition, air compressor, etc.)

In the simplest case $\mathbf{L}$ is constant; more generally, it is a periodic function of ψ with period 2π. We assume $\mathbf{L}$ to be given by

$$\mathbf{L} = \mathbf{L}_m[1 + w(\psi)] \quad \text{with} \quad \int_0^{2\pi} w(\psi)\, d\psi = 0, \tag{2}$$

in which w is a periodic function and $\mathbf{L}_m$ is the mean load resistance. The operation is steady only if the net work done per period is zero:

$$\int_0^{4\pi} (\mathbf{M}^{(n)} - \mathbf{L})\, d\psi = 0,$$

which, together with (2) and (7, 3), leads to the following condition for steady operation:

$$\mathbf{L}_m = \mathbf{M}_m^{(n)}. \tag{3}$$

Thus L_m must equal the average ordinate of the curve $M^{(n)}$, obtained by planimetering the area contained between the curve $M^{(n)}$ and the ψ-axis.

If we plot the useful torque $M^{(n)}$ together with the load resistance L as functions of ψ (Fig. 20), the difference gives the torque surplus-deficit for the engine-load combination considered:

$$U = M^{(n)} - L. \tag{4}$$

If the coefficient of cyclic variation ε (7, 10) is specified, we can easily find from the curve U the value of the flywheel moment of inertia A required by the engine. The equation of motion of the flywheel is

$$A\dot{\omega} = \frac{1}{2}A\frac{d(\omega^2)}{d\psi} = U. \tag{5}$$

According to this equation, the angular velocity ω has maxima and minima at those angles of rotation of the crank ψ for which $U = 0$, i.e. at those

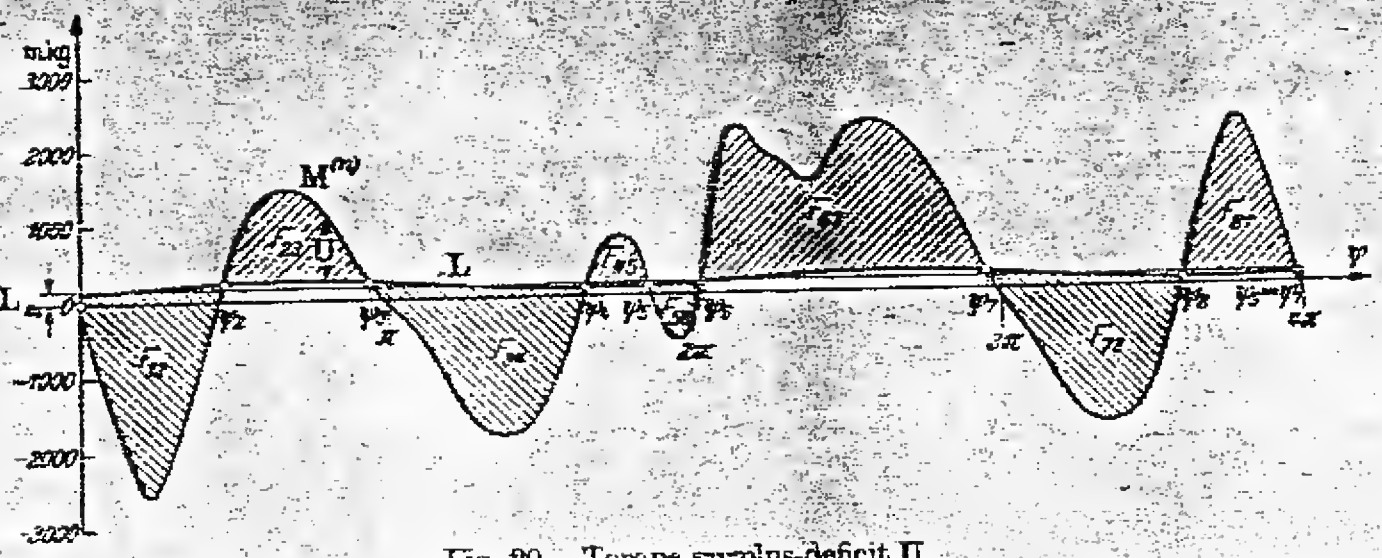


Fig. 20.—Torque surplus-deficit U.

points of Fig. 20 where the curves for $M^{(n)}$ and L cross. Moreover, where $M^{(n)}$ lies above L the angular velocity is increasing, and vice versa. According to (5), the areas of the individual cross-hatched positive and negative portions of Fig. 20 are

$$F_{i,i+1} \equiv \int_{\psi_i}^{\psi_{i+1}} U\,d\psi = \tfrac{1}{2}A(\omega_{i+1}^2 - \omega_i^2), \tag{6}$$

giving, with the factor $A/2$, the differences of the squares of consecutive extreme values of ω. According to (7, 9), with given numbers $\varepsilon_i \leqq \varepsilon/2$ and $\varepsilon_{i+1} \leqq \varepsilon/2$, we can write

$$\omega_i = \bar{\omega}(1 \pm \varepsilon_i), \qquad \omega_{i+1} = \bar{\omega}(1 \mp \varepsilon_{i+1}),$$

in which the upper algebraic signs are used if ω_i is a maximum and ω_{i+1} a minimum. The lower signs are used in the opposite case. Then

$$\tfrac{1}{2}(\omega_{i+1} + \omega_i) = \bar{\omega}(1 + \tfrac{1}{2}\varepsilon_{i,i+1}) \quad \text{with} \quad \varepsilon_{i,i+1} = \pm(\varepsilon_i - \varepsilon_{i+1}).$$

Since $|\varepsilon_{i,i+1}| \leqq \varepsilon/2$, we find, accurately enough, for small ε

$$\tfrac{1}{2}(\omega_{i+1} + \omega_i) = \bar{\omega}, \tag{7}$$

getting, instead of (6),

$$F_{i,i+1} = A\bar{\omega}(\omega_{i+1} - \omega_i). \tag{8}$$

Next, we planimeter the shaded areas $F_{i,i+1}$ and calculate the series of quantities:

$$F_{12},\ F_{13} = F_{12} + F_{23},\ F_{14} = F_{13} + F_{34},\ F_{15} = F_{14} + F_{45},\ \ldots,\ F_{1,n+1} = F_{1,n} + F_{n+1} \tag{9}$$

in which, as a test of accuracy, $F_{1,n+1} = 0$, since over the whole period the surplus and deficit torques balance. We then pick out the largest and the smallest terms in the set ($F_{1k} \equiv F_{\max}$, $F_{1l} \equiv F_{\min}$), and with (8) and (7, 10) obtain the expression

$$F_{\max} - F_{\min} = A\bar{\omega}(\omega_{\max} - \omega_{\min}) = A\bar{\omega}^2\varepsilon$$

or

$$A = \frac{F_{\max} - F_{\min}}{\bar{\omega}^2\varepsilon}, \tag{10}$$

in which ω_m^2 can replace $\bar{\omega}^2$, according to (7, 13). In this way, A, the necessary flywheel moment of inertia, is found. (In numerical calculation it is necessary, of course, that the areas be expressed in units of dimension $\mathbf{M}^{(n)}\psi$).

By this calculation we somewhat overestimate the required moment of inertia, usually a safe thing to do. Actually, the various moving masses of the engine are equivalent to an additional energy reservoir. Since we have not considered the ω-terms of $\mathbf{M}''$ in the $\mathbf{M}^{(n)}$-curve or in $\mathbf{U}$ (i.e. $\frac{1}{2}\Psi d(\omega^2)/d\varphi$ of (4, 7)) we must write instead of (5)

$$\frac{\mathbf{I}}{2}(A + \Psi)\frac{d(\omega^2)}{d\varphi} = \mathbf{U}. \tag{11}$$

For the first approximation we replace Ψ by its average value

$$\Psi_m = \frac{1}{2\pi}\int_0^{2\pi} \Psi\, d\varphi \tag{12}$$

and get instead of (10)

$$A = \frac{F_{\max} - F_{\min}}{\bar{\omega}^2\varepsilon} - \Psi_m. \tag{13}$$

If we must be very careful not to underestimate A—a possibility if (13) is applied to engines with widely varying Ψ, as happens in a compact design—we may prefer to use $\Psi_{\min}$ instead of Ψ_m, getting

$$A = \frac{F_{\max} - F_{\min}}{\bar{\omega}^2\varepsilon} - \Psi_{\min}. \tag{14}$$

We get a still better value of A if we write (11) in the form

$$\frac{1}{2}A\frac{d(\omega^2)}{d\varphi} = \mathbf{U} - \frac{1}{2}\Psi\frac{d(\omega^2)}{d\varphi},$$

inserting in the correction term $\frac{1}{2}\Psi d(\omega^2)/d\varphi$ the approximate value $d(\omega^2)/d\varphi = \mathbf{U}/\frac{1}{2}A_0$ obtained from (5), where A_0 is a good approximation for A, e.g. expression (10). Then we get

$$\frac{1}{2}A\frac{d(\omega^2)}{d\varphi} = \mathbf{U}\left(1 - \frac{\Psi}{A_0}\right) \equiv \mathbf{U}^*. \tag{15}$$

Thus we multiply all the ordinates of the U-diagram by the factor $(1 - \Psi/A_0)$ obtained directly from Fig. 7 of Sect. 4. Next we determine the areas $F^*_{i,i+1}$ corresponding to the corrected ordinates $\mathbf{U}^*$, getting, as before, the series F^*_{12}, F^*_{13}, F^*_{14}, etc., Finally, the expression

$$A = \frac{F^*_{\max} - F^*_{\min}}{\bar{\omega}^2\varepsilon} \tag{16}$$

gives the required moment of inertia of the flywheel with considerable accuracy.

For the example illustrated in Fig. 20, with $\omega = 20\pi$ and $\varepsilon = 0{\cdot}01$, the formulæ (10), (13), (14) and (16) give the following values for A: 110·0; 108·0; 108·5; 106·3 m.kg.sec.^{2} (corresponding to a flywheel weighing about 3000 kg. and with radius of gyration equal to 0·6 m.). As shown by this example, in general the simple formula (10) is perfectly satisfactory in practice for flywheel computation.

If the periods of $\mathbf{M}^{(n)}$ and $\mathbf{L}$ are appreciably different, i.e. if the load has a period other than 2π or 4π, it is necessary to plot both $\mathbf{M}^{(n)}$ and $\mathbf{L}$ for enough engine revolutions for repetition to occur. The analysis is then based on this extended curve. These remarks apply also to the topics that are taken up in the following paragraphs.

8. **The Operation of the Engine; First Method.**—When the flywheel has been calculated or is prescribed by other considerations (e.g. a propeller), the operation of the engine, i.e. ω as a function of φ and φ as function of t, is quickly determined. For a first approximation we start with (8, 5) and, introducing the dimensionless variable

$$z = \frac{\omega^2}{\omega_m^2}, \tag{1}$$

first obtain

$$\frac{dz}{d\varphi} = \frac{2\mathbf{U}}{A\omega_m^2}, \tag{2}$$

from which, with the (unknown) initial boundary condition $z_0 = \omega_0^2/\omega_m^2$ for $\varphi = 0$, there follows

$$z = z_0 + \frac{2}{A\omega^2}\int_0^\varphi \mathbf{U}\,d\varphi. \tag{3}$$

According to Sect. 8 the terms of U that depend on ω can be expressed in terms of ω_m^2. The second term on the right-hand side of (3) is obtained by plotting the torque surplus-deficit curve U as function of φ, integrating it graphically. If the load resistance L is constant ($= L_m$) it is not necessary to construct the U-curve anew but only to move the φ-axis in Fig. 20 vertically by the amount L_m. In Fig. 21 this axis is indicated by the symbol (φ) (whereas the axis designated by φ is not used).

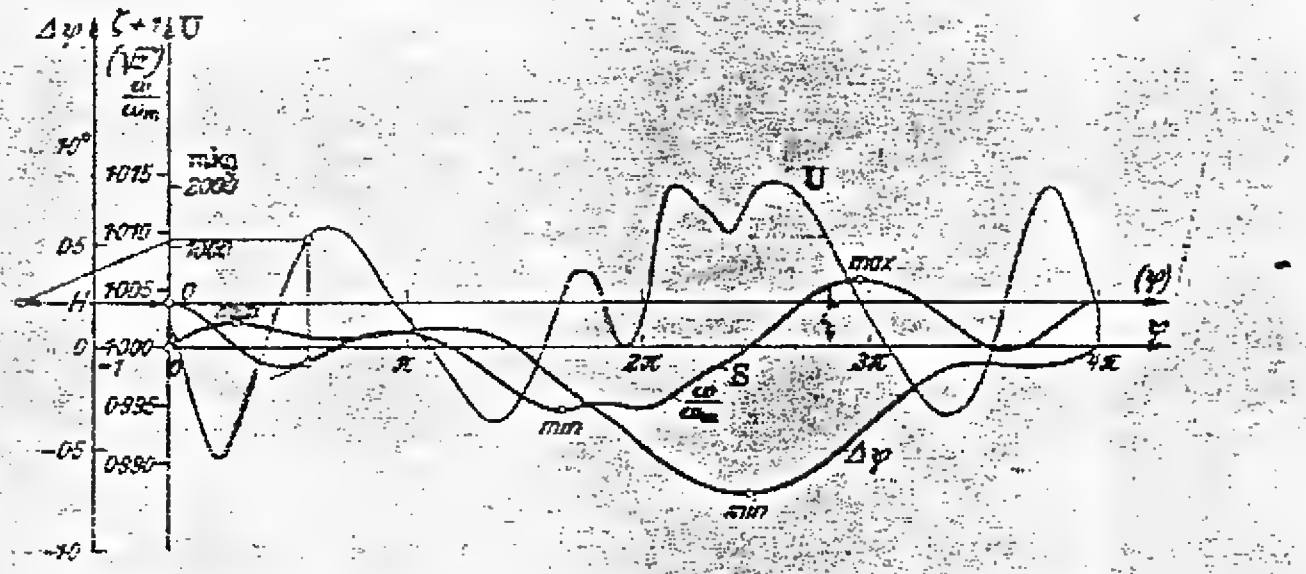


Fig. 21.—Operation of the engine: speed deviation z and the angular deviation $\Delta\varphi$.

The integral of U can be obtained graphically by constructing a curve whose tangent at any point φ has the slope U/H, H being the pole distance used in the construction. The result is an integral curve with ordinates that are $1/H$ the correct ordinates. Making

$$H = \frac{A\,\omega_m^2}{2m}, \tag{4}$$

where m is a factor sufficient to cause H to lie between 1 and 2, we get a curve whose ordinates are m times the quantity

$$S = \frac{2}{A\,\omega_m^2}\int_0^{\varphi} U\,d\varphi. \tag{5}$$

Fig. 21 illustrates the construction for $A = 100$ m.kg.sec.^{2}, $\omega_m = 20\pi$ sec.$^{-1}$ and $m = 10^5$, giving the curve S. As a check, this curve must again become zero when $\varphi = 4\pi$. Its ordinates represent the quantity $z - z_0$ (apart from the scale factor m). If we take the average ordinate of this curve and designate it $z = 1$ (or $\omega = \omega_m$) we have only exchanged the mean value $(\omega^2)_m$ of ω^2 for the square of the actual mean value ω_m, which is quite permissible, according to (7, 15). The φ-axis is placed at $z = 1$, so that, according to (3), the new ordinates of the integral curve give the values of $z = \omega^2/\omega_m^2$

multiplied by m. Therefore, the units of the z-axis are m times the units of $\mathfrak{U}$.

Next, we must take the square root of all the ordinates z of the integral curve. Since these ordinates are differing slightly from unity, they can be written in the form $z = 1 + 2\zeta$, where $|\zeta|$ is a very small number $< \varepsilon$, so that it is a very good approximation to write $\omega/\omega_m \equiv \sqrt{z} = 1 + \zeta$. Thus, ζ must be the speed deviation defined in (7, 18). Consequently, instead of extracting the square roots of the ordinates of the integral curve and replotting a new curve, it is possible simply to change the scale of the ordinate axis, making the new unit twice the old, or $2m$ times the unit of $\mathfrak{U}$, obtaining from the $\mathfrak{S}$-curve and the new ordinates the values of ω/ω_m as functions of the crank position φ (with an uncertainty usually less than the thickness of the curve). Fig. 21 illustrates this construction. The deviation ζ of the ω/ω_m-curve from the φ-axis gives the speed deviation directly. The coefficient of cyclic variation ε can also be obtained from the curve; here

$$\varepsilon = \frac{\omega_{\max}}{\omega_m} - \frac{\omega_{\min}}{\omega_m} = 1{\cdot}0067 - 0{\cdot}9945 = 0{\cdot}0122,$$

which is somewhat larger than in the example of Sect. 8, as would be expected on account of the smaller moment of inertia of the flywheel.

This simple construction, which amounts to nothing more than a graphical integration, an averaging process, and a change of scale, can be very easily extended to give the angular deviation $\Delta\varphi$ of the engine. According to (7, 19) $\Delta\varphi$ is obtained by integrating over the speed deviation ζ, i.e. by integrating the ω/ω_m-curve graphically with the φ-axis as base. If the unit distance along the φ-axis (scale) is taken as the pole distance, the unit for $\Delta\varphi$ is the same as that for ω/ω_m, i.e. the interval corresponding to 0·01 on the ω/ω_m-axis corresponds to an angular deviation of $0{\cdot}01 \times 180°/\pi = 0{\cdot}573°$. A check on the accuracy of the computation is that if $\Delta\varphi$ is zero at $\varphi = 0$ it must again be zero at $\varphi = 4\pi$. In Fig. 21 we find the maximum angular deviation to be 0·85°.

We have not yet obtained φ as a function of t, but will have no difficulty in getting that relation. Since $\Delta\varphi = \varphi - \omega_m t$ or $\omega_m t = \varphi - \Delta\varphi$, for each crank position we have the value of $\omega_m t$ as function of φ and therefore can construct a (φ, t)-diagram immediately. However, with the very small values of $\Delta\varphi$ that actually occur this relation would be barely distinguishable from a straight line; consequently, the abscissæ φ that belong to $\Delta\varphi$ in Fig. 21 may also be taken as equal to $\omega_m t$ without appreciable error. Then

$$\varphi = \omega_m t + \Delta\varphi(t) \tag{6}$$

gives the crank position as a function of time. Thus, this last problem, which is seldom important, is solved.

The whole process can be progressively improved (if that is necessary) by replacing the equation of motion (8, 3) by the more exact equation (8, 11), so that we write instead of (2)

$$\frac{dz}{d\psi} = \frac{2U}{(A + \Psi)\omega_m^2}. \tag{7}$$

In this approximation Ψ may be replaced by its mean value Ψ_m (8, 12) and the pole distance H (4) by

$$H' = \frac{(A + \Psi_m)\omega_m^2}{2m}. \tag{8}$$

If the integral curve based on the pole distance H has already been constructed, the improved curve can be obtained simply by changing the ordinate scale so that the new unit is H'/H times the old.

We get a still more accurate result if in (7) we also take account of the variation in Ψ, getting the equation

$$\frac{dz}{d\psi} = \frac{2}{A\omega_m^2} \frac{U}{1 + \frac{\Psi}{A}},$$

which may be rewritten accurately enough as

$$\frac{dz}{d\psi} = \frac{2}{A\omega_m^2} U\left(1 - \frac{\Psi}{A}\right) = \frac{2U^*}{A\omega_m^2}. \tag{9}$$

This involves carrying through the construction with the new improved relation U^* instead of U.

In the example of Fig. 21 both improvements give effects which are of the same order as the unavoidable graphical errors. In general, for normal values of ε and A the first approximation (2) is quite adequate. This is no longer true with engines that run very unsteadily, so that in the torque M'' (Sect. 4) neither is the term containing $\dot{\omega}$ negligible nor is ω^2 replaceable by ω_m^2. In this case the following method is suitable.

10. The Operation of the Engine; Second Method.—We start with the exact equation of motion of the flywheel, which, with the help of (2, 3) and (4, 2), we write in the form

$$A\dot{\omega} \equiv \frac{1}{2} A \frac{d(\omega^2)}{d\psi} = M^{(a)} - L \equiv M' - \frac{1}{2}\frac{d}{d\psi}(\Psi\omega^2) - M''' - M'''' - L$$

or

$$\frac{d}{d\psi}[(A + \Psi)z] = \frac{2}{\omega_m^2}(M^{(e)} - L) \quad \text{with} \quad z = \frac{\omega^2}{\omega_m^2}. \tag{1}$$

The effective torque is

$$M^{(e)} = M' - (M''' + M''''). \tag{2}$$

We easily get the $\mathbf{M}^{(e)}$-curve (Fig. 22) from the curve for the tangential force torque $\mathbf{M}'$ and those for $\mathbf{M}'''$ and $\mathbf{M}''''$ (Fig. 19 of Sect. 8). We may unhesitatingly base the friction torque $\mathbf{M}''''$ on the average angular velocity ω_m, particularly since this torque is relatively small and its value not precisely known in any case. The effective torque $\mathbf{M}^{(e)}$ is represented in the form of a Fourier series whose coefficients differ only very slightly from the coefficients a_j, b_j of the Fourier development of the tangential force torque $\mathbf{M}'$ (3, 3) or (3, 4) since the difference between $\mathbf{M}^{(e)}$ and $\mathbf{M}'$ must be small according to (2). On the other hand, in high-speed engines the effective torque $\mathbf{M}^{(e)}$ differs greatly from the useful torque $\mathbf{M}^{(n)}$ (Figs. 19 and 22). The effective torque $\mathbf{M}^{(e)}$ must always be used in place of $\mathbf{M}'$ in the equation of motion of an engine if it is desired to take account of gravity and friction, as will be done in Part III.

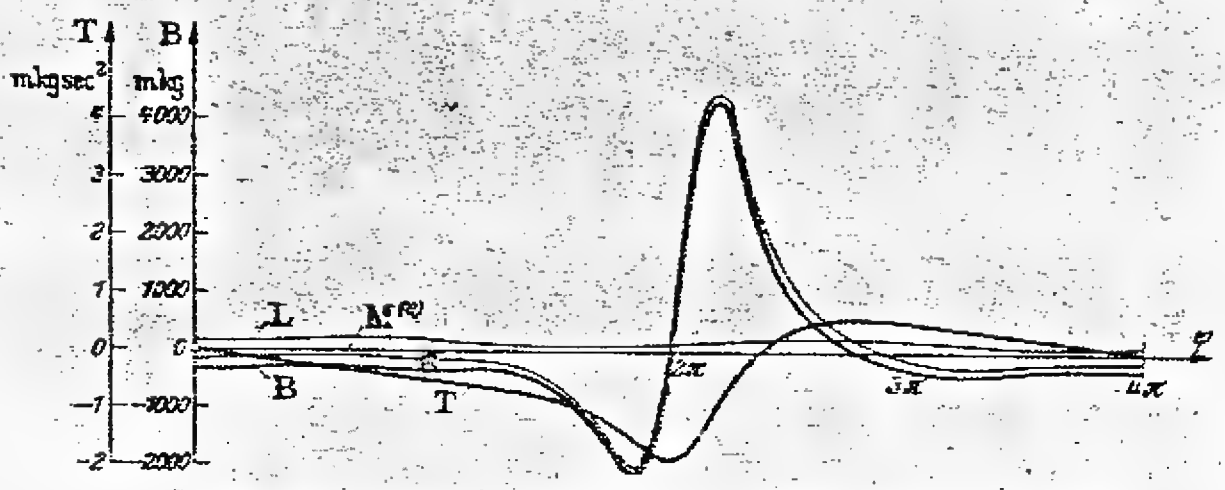


Fig. 22.—The effective torque $\mathbf{M}^{(e)}$, its surplus-deficit B and the integral of the surplus-deficit T.

We again introduce the torque of the load resistance L and write the equation for the effective torque surplus and deficit:

$$\mathbf{B} = \mathbf{M}^{(e)} - \mathbf{L}. \tag{3}$$

which is integrated to give

$$\mathbf{T} = \frac{2}{\omega_m^2} \int_0^\varphi \mathbf{B}\, d\varphi. \tag{4}$$

If this is evaluated graphically, using the pole distance

$$H = \frac{\omega_m^2}{2m}, \tag{5}$$

the ordinates of T will be in units that are m times the units of B. In Fig. 22 $\omega_m = 20\pi$ sec.$^{-1}$ and $m = 1000$.

By integrating equation (1) we get

$$(\Delta + \Psi)z = (\Delta + \Psi_0)z_0 + \mathbf{T}, \tag{6}$$

which is more accurate than the corresponding equation (9, 3) for engines with very uneven motion. There are two problems that can be solved by means of this equation: finding the necessary moment of inertia A of the flywheel when the coefficient of cyclic variation ε (or ε') is given; if A is given, finding the speed variation of the engine. In each case z_0 is unknown while Ψ and T are known functions of ψ (Fig. 7 of Sect. 4 and Fig. 22); Ψ_0 is the value of Ψ at $\psi = 0$.

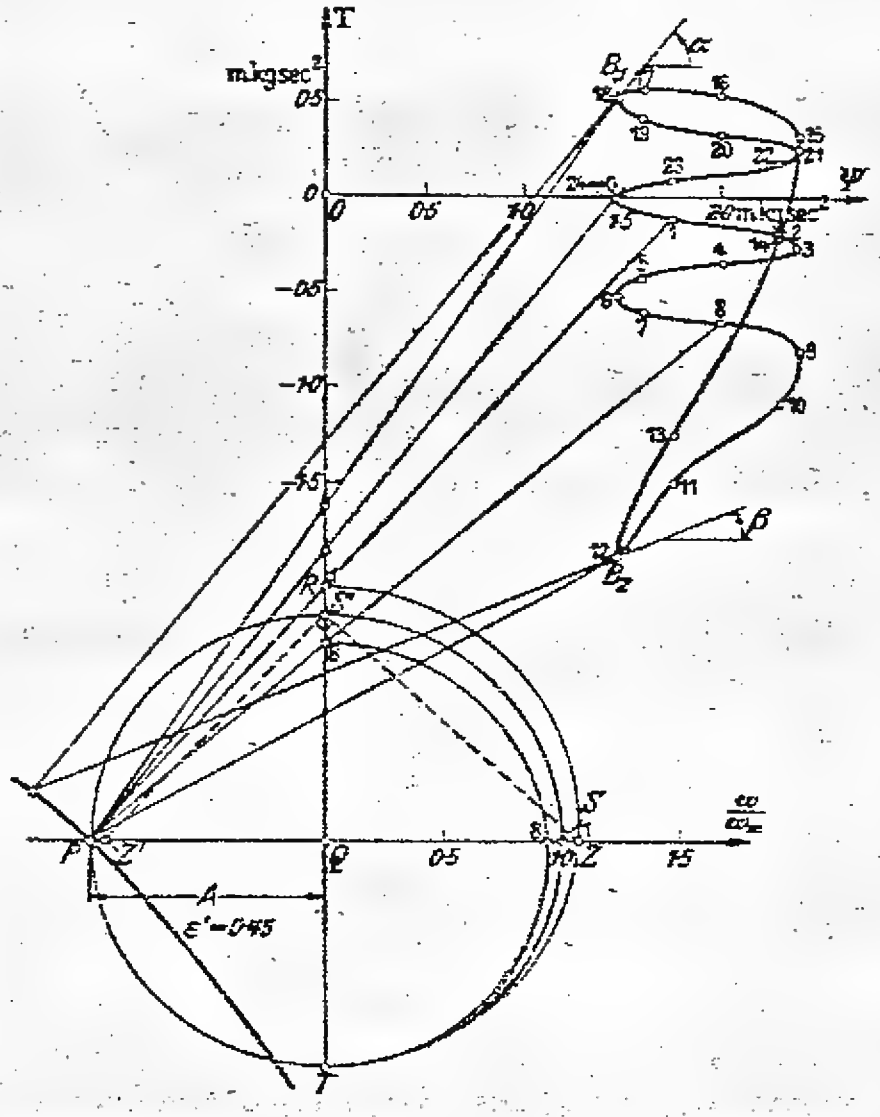


Fig. 23.—Graphical determination of ω/ω_m from T and Ψ.

We shall solve the second problem first and in order to do so we construct Fig. 23 [1] in which are plotted the values of Ψ and T (as coordinates) that correspond to each crank position ψ. The result is a closed curve on which are marked (during construction) points corresponding to equally spaced values of the parameter ψ. In Fig. 23 these points are numbered 0–24 for $\psi = 0$ to $\psi = 4\pi$.

If z_0 (for $\psi = 0$) were known in addition to A it would only be necessary to plot the point P, with coordinates $-A$, $-(A + \Psi_0)z_0$ in order to have the operation of the engine immediately. If we connect a point (Ψ, T) of

[1] F. Wittenbauer, "Graphische Dynamik", Berlin 1923; p. 759. (Also see the bibliography given on p. 775 of this reference.)

the diagram (for example, point 1 in Fig. 23) with point P, then according to equation (6),

$$\frac{QR}{PQ} = \frac{(A + \Psi_0)z_0 + \mathbf{T}}{A + \Psi} = z. \tag{7}$$

Since $PQ = A$ we have simply $QR = Az$. If we construct a circle about Q with radius A and then on RT as diameter construct a semicircle that meets the extension of PQ in Z, we have

$$QZ = \sqrt{TQ \,.\, QR} = A\sqrt{z} = A\,\frac{\omega}{\omega_m}. \tag{8}$$

Thus the value of ω/ω_m (with correction for the factor A) can be read directly from the diagram (the distance QZ) for each value of the parameter ψ. (Fig. 23 shows the construction at points 1 and 8 for $A = 1{\cdot}2$ m.kg.sec.^{2})

This simple procedure is hindered only by the fact that actually the length of $QO = (A + \Psi_0)z_0$ is at first unknown. We get it, however, from the consideration that the mean value of the quantities $1/\omega$ must be equal to $1/\omega_m$ (7, 2), so that the mean value of the ratio QS/QZ ($\equiv \omega_m/\omega$) is unity. Thus, the location of the point Q can be found by trial and error, the condition being that the fan of lines from P to the points 0, 1, 2, . . . of the $\mathbf{T}$, Ψ-curve is grouped more or less uniformly on both sides of the ray PS' so that S' (a distance A above Q) is approximately the centre of gravity of the set of points R. Then S is likewise approximately the centre of gravity of the points Z, so that the mean value of $1/\omega$ is certainly close to $1/\omega_m$. It is necessary to repeat the whole construction with different lengths OQ until the location for Q is found with enough accuracy, or can be found by interpolation. If we start with a well-chosen first approximation for Q, the final result is usually reached in only two steps. The averaging of the ratios QS/QZ can be carried out graphically by drawing ZS' and $S'Z'$ perpendicular to ZS' and then getting the centre of gravity of the points Z' in the usual way: it must coincide with P if Q is correct. In general, it will be quicker to calculate values of the ratio QS/QZ by slide-rule (for constant QS) and then to find the average of these values: it must equal unity.

Finally, in Fig. 24 we plot the intervals SZ (or convenient multiples thereof) as ordinates above the ψ-axis (which is displaced a unit distance vertically) and thus get the desired graph of the operation of the engine as well as the coefficient of cyclic variation ε or ε'. In the present example $\varepsilon = 0{\cdot}46$ and $\varepsilon' = 0{\cdot}45$. This unusually large coefficient results from the very small moment of inertia $A = 1{\cdot}2$ m.kg.sec.^{2}, as does the noticeable difference between ε and ε'.

From the curve for ω/ω_m we can find the angular deviation:

$$\Delta\psi \equiv \psi - \omega_m t = \psi - \omega_m \int_0^{\psi} \frac{d\psi}{\omega} = \int_0^{\psi} \left(1 - \frac{\omega_m}{\omega}\right) d\psi. \tag{9}$$

The curve $(1 - \omega_m/\omega)$ can be plotted—it is not shown in Fig. 24—and integrated graphically to yield $\Delta\psi$. In this example $\Delta\psi_{max} - \Delta\psi_{min} = 37°$, a very large difference.

It must be mentioned that this construction, like all operations made on the diagram in question, becomes awkward as soon as A is large enough

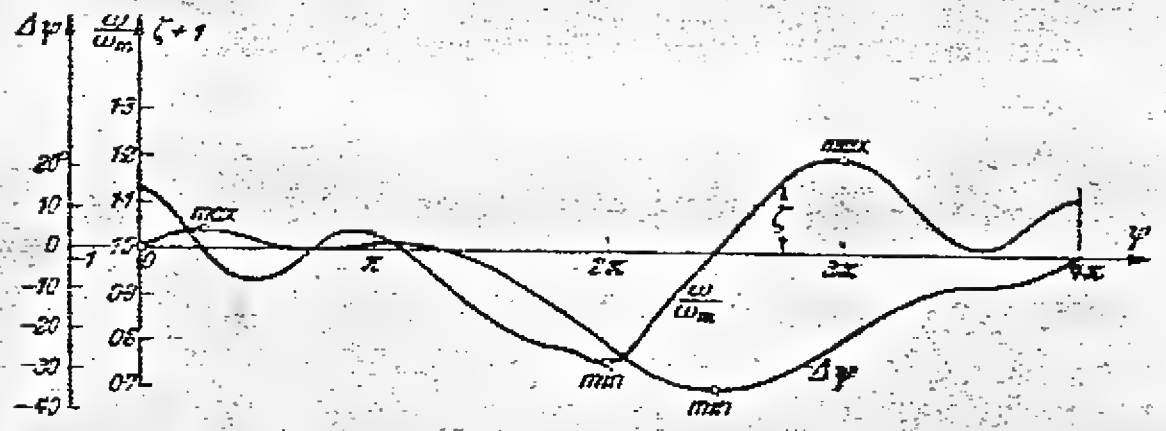


Fig. 24.—Operation of the engine: speed deviation ζ and the angular deviation $\Delta\psi$.

to cause the point P to lie at a great distance from the $\mathfrak{T}$, Ψ-curve. The construction becomes easier again when A has become so large that Ψ_0 is small by comparison. In this case ε is also small and since $|z_0 - 1| < \varepsilon$, $z_0 = 1$ nearly. This means that the rays PR in Fig. 23 are nearly parallel and at approximately 45° with the coordinate axes, so that the points R can be found without reference to P simply by drawing lines with 45° slope from the points 0, 1, 2, . . . of the curve (Fig. 25). Since again $(\omega^2)_m = \omega_m^2$ nearly, it is only necessary to find the centre of gravity S' of the series of points R. The values of ω/ω_m can be read off the $\mathfrak{T}$-axis immediately (at the corresponding points R). In doing this we assign the ordinate unity to the point S' and use units that are $2A$ times the original $\mathfrak{T}$-ordinate units. (The factor 2 is due to the change from ω^2/ω_m^2 to ω/ω_m.)

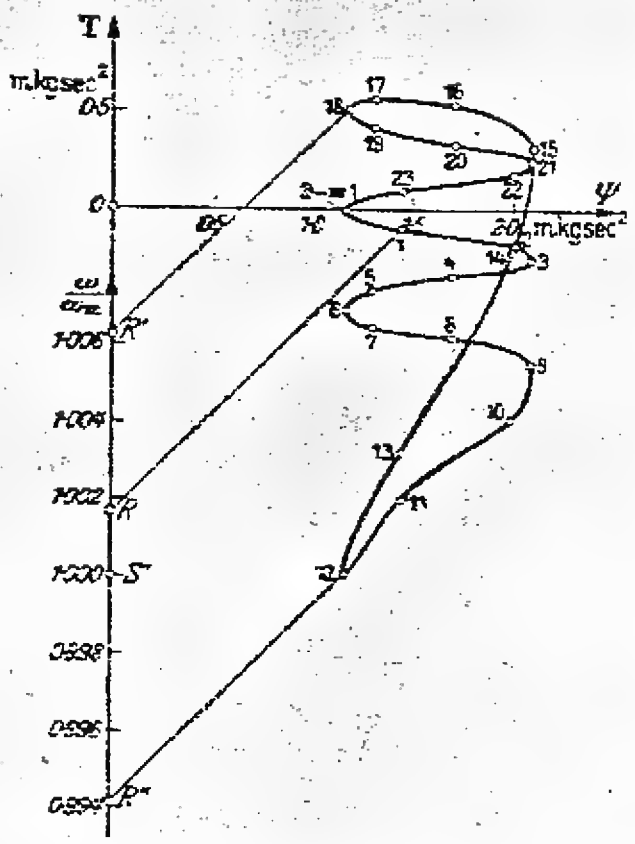


Fig. 25.—Graphical determination of ω/ω_m from $\mathfrak{T}$ and Ψ for large A.

The next steps are like those in the first method (Sect. 9). The coefficient of cyclic variation ε can be read directly from the ω/ω_m-axis as the difference between the ordinates of the highest and lowest 45° tangents to the $\mathfrak{T}$, Ψ-curve, expressed in terms of the ω/ω_m units. In the example of Fig. 25, where $A = 100$ m.kg.sec.², $\varepsilon = 0{\cdot}0121$, which agrees well with the result of Sect. 9.

The second method, in this approximation, is naturally no more accurate

than the first; since the second method is generally somewhat more laborious, the first method is usually preferred and the second is used only for very small values of A, for which the first method is not sufficiently exact.

11. **Flywheel Calculation; Second Method.** — The "second" problem, which has just been solved, can be characterized by the statement that a point P is to be found which lies on a line parallel to the T-axis and a distance A from it, and which corresponds to a specified value of ω_m. The "first" problem, now to be solved, is one in which the coefficient of cyclic variation ε or ε' is given, and the inertia moment of the flywheel A is to be found, as well as the operation of the engine. This differs from the "second" problem only in that now the geometric path of P is another curve, corresponding to a given value of ε or ε'.

In order to locate points on this curve we remember that according to (**10**, 7) the inclination α of any line connecting a point ψ of the T, Ψ-curve with P is given by the expression $\tan\alpha = z\,(=\omega^2/\omega_m^2)$, where ω is the angular velocity corresponding to the crank rotation ψ. This relation holds, in particular, for the highest and the lowest tangents drawn from P. Therefore, if α and β are the slopes of these tangents (Fig. 23),

$$\sqrt{\tan\alpha} - \sqrt{\tan\beta} = \frac{\omega_{max} - \omega_{min}}{\omega_m} = \varepsilon'. \qquad (1)$$

If β is given an arbitrary value, the corresponding value of α for the prescribed ε' follows from (1), resulting in a pair of tangents whose intersection determines a point P. If this construction is repeated for other values of β the intersections define a curve, the ε'-curve, which is the geometrical path of P. This is shown in Fig. 23 for $\varepsilon' = 0{\cdot}45$.

The rest of the construction proceeds exactly as in the "second" problem: by trial and error we find a point P on the ε'-curve such that the rays to the T, Ψ-curve are more or less uniformly distributed about S', and then find the corresponding average value ω_m. This process is repeated for different locations of P (all on the ε'-curve) until the prescribed value of ω_m is obtained. Then PQ, the distance of P from the T-axis, gives the necessary flywheel moment of inertia A. The operation of the engine is then found exactly as in the "second" problem.

Since ε and ε' may be interchanged when they are small, and since usually only ε' is specified when they are large, it is not necessary to consider what to do if ε is given instead of ε'.

On the other hand, it must be mentioned that the actual construction of the ε'-curve is very difficult and rather inexact on account of the acuteness of the intersections of the tangents, which reduces the practical value of the method greatly. In the usual case, where the tangent points B_1 and B_2 (Fig. 23) lie nearly one above the other and the T, Ψ-curve bends so sharply in their neighbourhood that the points do not move appreciably

with changes in α and β, a simple construction is possible for approximating to the ε'-curve. In this approximation the two tangent points (very nearly the points 12 and 18 of Fig. 23) are kept fixed. Then, with the help of Fig. 26 in which $a = B_1B_2$ (the fixed distance between the extreme tangent points), we find from (1) an equation for the path of point P in an x, y-co-ordinate system whose origin is at the lower tangent point B_2:

$$\sqrt{\frac{a+y}{x}} - \sqrt{\frac{y}{x}} = \varepsilon',$$

so that

$$y = \frac{a^2 + \varepsilon'^2 x(\varepsilon'^2 x - 2a)}{4\varepsilon'^2 x}. \qquad (2)$$

Thus, the ε'-curve is approximately a hyperbola; it is quickly calculated for given a and ε', and is shown in Fig. 23.

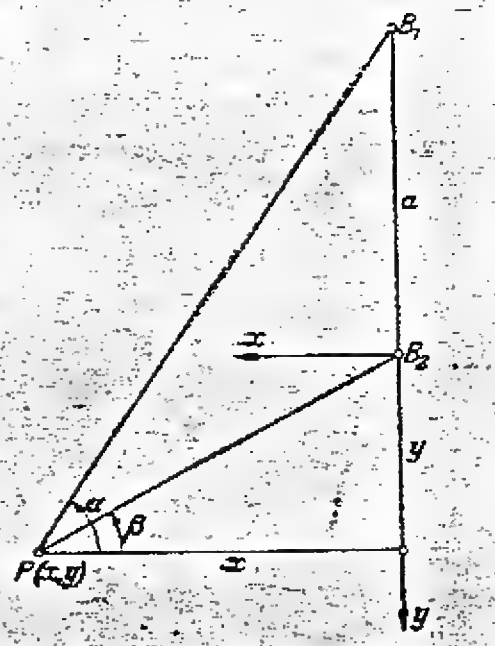


Fig. 26.—Approximate method of finding the ε'-curve.

For small values of ε' the hyperbola moves away to a great distance, whereupon the construction again becomes very simple. In Fig. 25 the length $R'R''$, measured not in the ω/ω_m units (as yet unknown) but in the original T-units, according to (10, 7), amounts to

$$R'R'' = A\left(\frac{\omega_{max}^2}{\omega_m^2} - \frac{\omega_{min}^2}{\omega_m^2}\right) = A\,\frac{\omega_{max} + \omega_{min}}{\omega_m}\,\frac{\omega_{max} - \omega_{min}}{\omega_m}.$$

Since $\omega_m = \bar{\omega} = \frac{1}{2}(\omega_{max} + \omega_{min})$ approximately, the moment of inertia of the flywheel (A) is found from the simple formula

$$A = \frac{R'R''}{2\varepsilon} \qquad (R'R'' \text{ in T-units}). \qquad (3)$$

Again it must be mentioned that this value is less exact than that given by the first method, even though the present method generally involves more labour. Therefore, in solving the "first" problem, the present method should also be limited to the case of every non-uniform operation. It appears that apart from this rather rare situation the "first" method (Sect. 8 and 9) always has the advantage since it is generally simpler and practically always sufficiently exact.

12. The Offset Reciprocating Mechanism.—As already mentioned in I, 14, in single-acting engines there are definite advantages (although gained at the cost of the inertia-force balance) when the cylinder is offset with respect to the crank by a distance b (generally smaller than the crank throw r) so that the angle χ between the cylinder axis and the connecting-rod is smaller during the firing stroke than it would be with no offset (Fig. 29 of I, 14

or Fig. 27 of this Part). In attacking this problem we shall be able to make use of previously obtained results. For this, it is obviously only necessary to indicate how the determination of the torques $\mathbf{M}'$, $\mathbf{M}''$, $\mathbf{M}'''$ and $\mathbf{M}''''$ is changed by the existence of the offset.

Fig. 27 shows how the tangential force torque $\mathbf{M}'$ is found. The line of stroke is a distance b from the centre of the crank circle. We first locate the two dead-centres B_0 and B_0' of the gudgeon-pin (or cross-head) which are characterized by the fact that the centre lines OA_0 and A_0B_0 of the crank web and the connecting-rod (as well as OA_0' and $A_0'B_0'$) lie on a straight

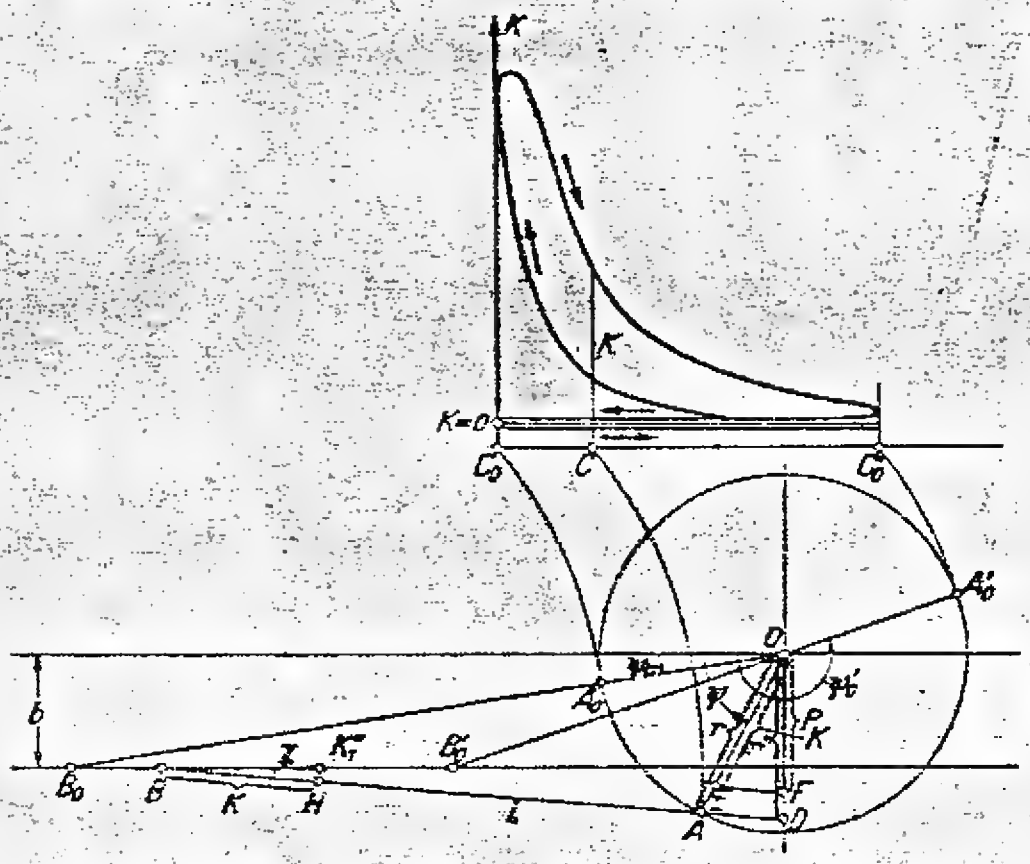


Fig. 27.—Graphical determination of the tangential crank force P from the piston force K for an offset reciprocating mechanism.

line. Then circles of radius l are drawn about B_0 and B_0', defining the points C_0 and C_0' through their intersection with a straight line drawn parallel to the line of stroke. The indicator diagram is placed above the base line C_0C_0' scaled so that the length of stroke equals the distance C_0C_0'. Since equations (3, 1) and (3, 2) also hold for offset reciprocating mechanisms the following steps are as in Sect. 3: for an arbitrary crank position φ a circular arc AC of radius l is drawn about B; the piston force K is measured directly above C and is then laid off to scale on OA from O to E. The line EF is drawn parallel to BA so that the distance OF gives the desired tangential force P. This force, multiplied by r, equals the torque $\mathbf{M}'$. Since the symmetry of the mechanism is destroyed by the offset it is necessary now to construct the whole crank circle in getting the complete tangential-force curve, instead of using a semicircle as before. When the construction is completed it will be found that for normal indicator diagrams even large amounts of offset

b ($< r$) affect the shape of the tangential-force curve hardly at all; the new curve is obtained almost exactly from that for a non-offset engine simply through a parallel displacement.

The torque corresponding to the kinetic energy is the same as for the non-offset engine since equations (4, 1) to (4, 3) are independent of whether or not the line of stroke intersects the crank axis. The only difference is that the curve for the effective moment of inertia (Ψ) is no longer symmetric. The Fourier series previously derived (4, 5) cannot be taken over; we shall not, however, take the trouble to write down the new form, which is easily found from I, 14.

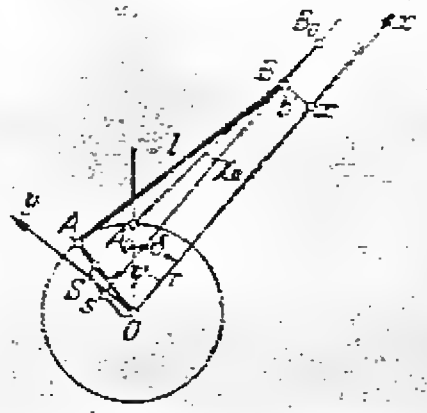


Fig. 28.—Offset reciprocating mechanism inclined from vertical.

From Fig. 28 (with the same symbols as in Fig. 8 of Sect. 5) we get the potential energy:

$$U = -\left[sG + r\left(1 - \frac{s'}{l}\right)G'\right][1 - \cos(\psi - \delta)] - \left(\frac{s'}{l}G' + G''\right)(r\cos\delta + l\cos\chi_0 - x)\cos\delta.$$

As before, $U = 0$ corresponds to the highest position A_0 of the crankpin A. Introducing the reduced first moments Q^* and $Q^{*\prime}$ (5, 1) as well as the expressions for x (I, 14, 3), we get for the torque corresponding to the potential energy,

$$\left.\begin{aligned}\mathfrak{M}''' \equiv \frac{dU}{d\psi} = -\Big[(Q^* + Q^{*\prime})\sin\psi + Q^*\Big(-A_2^*\cos\psi + \frac{1}{2}A_2^*\sin 2\psi + \\ + \frac{1}{3}A_3^*\cos 3\psi - \frac{1}{4}A_4^*\sin 4\psi - + + - \cdots\Big)\Big]\cos\delta + Q^{*\prime}\cos\psi\sin\delta.\end{aligned}\right\} \quad (1)$$

The relation with the steady longitudinal and transverse forces X^* and Y^* (I, 14, 9) is shown by rewriting equation (1) in the form

$$\mathfrak{M}''' = -\frac{g}{\omega^2}\left\{\cos\delta\left[\int_0^\psi X^*\,d\psi - Q^*\left(A_1^* - \frac{1}{3}A_3^* + \frac{1}{5}A_5^* - + \cdots\right)\right] + \sin\delta\int_{\pi/2}^{\psi} Y^*\,d\psi\right\}. \quad (2)$$

The torque $\mathfrak{M}'''$ vanishes only in horizontal engines ($\delta = \pm\pi/2$) with suitable inertia balancing. The special cases corresponding to Figs. 9 to 12 of Sect. 5 follow from equation (1) by putting $\delta = 0,\ \pi,\ \pi/2,\ -\pi/2$.

The internal forces in the offset mechanism are given directly by equations (6, 1) to (6, 8), (6, 11), (6, 12) and (6, 14). On the other hand, expressions (6, 9) and (6, 10) naturally assume new forms with the offset engine. If we limit ourselves again to first-order terms in $\lambda = r/l$ and in $\beta = b/l$ (the offset ratio), ignoring expressions containing λ^2, $\lambda\beta$, β^2 and higher powers, we get from (I, 14, 1) to (I, 14, 7),

$$\dot{x} = -r\omega(\sin\psi - \beta\cos\psi + \tfrac{1}{2}\lambda\sin 2\psi), \quad \ddot{x} = -r\omega^2(\cos\psi + \beta\sin\psi + \lambda\cos 2\psi). \quad (3)$$

$$\sin\chi = \lambda\sin\psi - \beta, \quad \cos\chi = 1, \quad \dot{\chi} = \lambda\omega\cos\psi, \quad \ddot{\chi} = -\lambda\omega^2\sin\psi. \quad (4)$$

If we again perform the entire calculation of Sect. 6, using these expressions, we get the bearing and connecting-rod forces K_1 to K_4 and P_1 to P_4, and the side-wall force K'':

$$
\left.
\begin{aligned}
K_1 &= K - G''\left[\frac{r\omega^2}{g}(\cos\psi + \underline{\beta}\sin\psi + \lambda\cos 2\psi) - \cos\delta\right] + \frac{\varepsilon'}{l}G'\sin\delta(\lambda\sin\psi - \underline{\beta}),\\
K_2 &= \frac{\varepsilon'}{l}G'\left[\sin\delta - \frac{r\omega^2}{g}\left(\frac{1}{2}\lambda\sin 2\psi - \underline{\beta\cos\psi}\right) + \cos\delta(\lambda\sin\psi - \underline{\beta})\right],\\
K_3 &= K - \left(\frac{\varepsilon'}{l}G' + G''\right)\left[\frac{r\omega^2}{g}(\cos\psi + \underline{\beta\sin\psi} + \lambda\cos 2\psi) - \cos\delta\right],\\
K_4 &= 0,\\
K'' &= K(\lambda\sin\psi - \beta) + \left(\frac{\varepsilon'}{l}G' + G''\right)\left[\sin\delta - \frac{r\omega^2}{g}\left(\frac{1}{2}\lambda\sin 2\psi - \underline{\beta\cos\psi}\right) + \right.\\
&\qquad \left. + \cos\delta(\lambda\sin\psi - \underline{\beta})\right],\\
P_1 &= K\left(\sin\psi - \underline{\beta\cos\psi} + \frac{1}{2}\lambda\sin 2\psi\right) + \left(1 - \frac{\varepsilon}{l}\right)G'\sin(\psi - \delta) -\\
&\quad - \left(\frac{\varepsilon'}{l}G' + G''\right)\left\{\frac{1}{2}\frac{r\omega^2}{g}\left[\sin 2\psi - \frac{1}{2}\lambda(\sin\psi - 3\sin 3\psi) - \underline{2\beta\cos 2\psi}\right] - \right.\\
&\quad \left. - \cos\delta\left(\sin\psi - \underline{\beta\cos\psi} + \frac{1}{2}\lambda\sin 2\psi\right)\right\},\\
P_2 &= K\left[\cos\psi + \underline{\beta\sin\psi} - \frac{1}{2}\lambda(1 - \cos 2\psi)\right] - \left(1 - \frac{\varepsilon}{l}\right)G'\left[\frac{r\omega^2}{g} - \cos(\psi - \delta)\right] -\\
&\quad - \left(\frac{\varepsilon'}{l}G' + G''\right)\left\{\frac{1}{2}\frac{r\omega^2}{g}\left[1 + \cos 2\psi + \frac{1}{2}\lambda(\cos\psi + 3\cos 3\psi) + \underline{2\beta\sin 2\psi}\right] - \right.\\
&\quad \left. - \cos\delta\left[\cos\psi - \underline{\beta\sin\psi} - \frac{1}{2}\lambda(1 - \cos 2\psi)\right]\right\},\\
P_3 &= P_1 + G\sin(\psi - \delta),\\
P_4 &= P_2 - G\left[\frac{r\omega^2}{g} - \cos(\psi - \delta)\right].
\end{aligned}
\right\} \qquad (5)
$$

The underlined terms in the expressions above (those containing β) are new ones, due to the offset. Most of them correspond to periodic forces whose average value is zero. One of the most important terms is $K(\lambda\sin\psi - \beta)$ in the first part of the expression for the side-wall force K''. The piston force is greatest during the working stroke $0 \leqq \psi \leqq \pi$, when $\lambda\sin\psi > 0$. Because of the presence of $-\beta$ this component $K(\lambda\sin\psi - \beta)$ is, on the average, very much diminished; since friction between piston and cylinder wall causes a significant part of the mechanical losses, an offset makes possible a substantial improvement in this respect—at least in low-speed engines where the static component of K'' is larger than the dynamic. This static component of K'' is

$$K_1'' = K(\lambda\sin\psi - \beta) = K\sin\chi. \qquad (6)$$

It can be obtained graphically as illustrated in Fig. 27: the piston force K is laid off from B towards A, locating the point H whose distance from the

line of stroke is the static component K_1''. Fig. 29 shows results based on the indicator diagram of Fig. 27 for three values of the ratio

$$\alpha = \frac{\beta}{\lambda} = \frac{b}{r}. \tag{7}$$

It is obvious that in this figure increasing the value of α results in diminished average static side-wall force K_1'' during the working stroke. Since the negative forces that occur during compression and at the beginning of burning must be treated as positive so far as friction is concerned, the performance deteriorates again for values of α exceeding $\frac{1}{4}$. In general, an offset that makes α lie between 1/8 and 1/4 is the best. Beyond $\alpha = 1/4$ there is the further disadvantage that during the compression stroke very large connecting-rod angles γ occur.

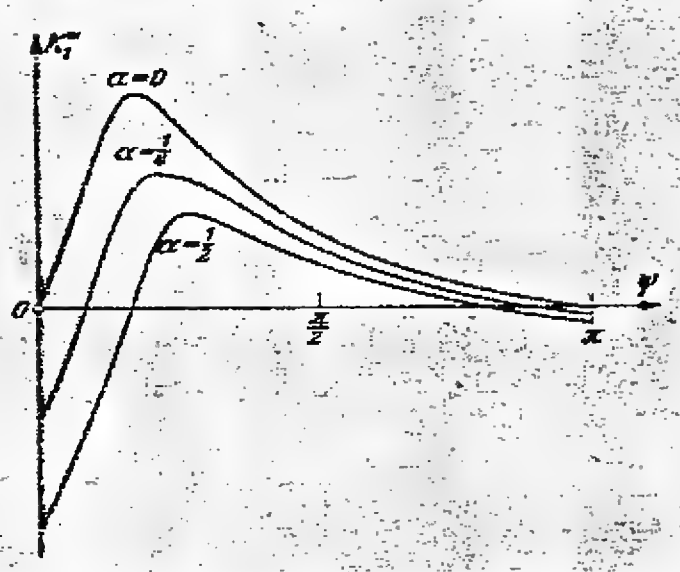


Fig. 29.—The static component of the cylinder-wall force K'' for different values of $\alpha = \beta/\lambda = b/r$.

These advantages of the offset engine are much less important in high-speed engines because the dynamic force component

$$\left(\frac{s'}{l}G' + G''\right)\frac{r\omega^2}{g}\left(\frac{1}{2}\lambda \sin 2\psi - \beta \cos \psi\right)$$

increases with speed, and is not diminished by the offset. Thus, offset is more suitable for low-speed than for high-speed engines.

The friction torque $\mathfrak{M}''''$ is calculated from the internal forces (5) exactly as in the standard mechanism. Similarly, the determination of the operation of the engine and finding the flywheel moment of inertia proceed exactly as before, after the useful torque $\mathfrak{M}^{(n)} = \mathfrak{M}' - (\mathfrak{M}'' + \mathfrak{M}''' + \mathfrak{M}'''')$ or the effective torque $\mathfrak{M}^{(e)} = \mathfrak{M}' - (\mathfrak{M}''' + \mathfrak{M}'''')$ is found.

§ 2. The Multi-cylinder Engine

13. The In-line Engine.—Carrying the previous methods over from the one-cylinder to the in-line engine is comparatively simple if, as is generally the case, all elements of the in-line engine are identical, i.e. all cranks, connecting-rods, cylinders, and even indicator diagrams.[1] (Small differences among indicator diagrams need not be taken into account since they are generally not exactly known.) We designate by ψ the position of the "first" crank, for example, the one farthest from the flywheel, and for this mechanism obtain all needed diagrams, i.e. $\mathfrak{M}_1'$, $\mathfrak{M}_1''$, $\mathfrak{M}_1'''$ and $\mathfrak{M}_1''''$, and from them, according to need, either the useful torque $\mathfrak{M}_1^{(n)} = \mathfrak{M}_1' - (\mathfrak{M}_1'' + \mathfrak{M}_1''' + \mathfrak{M}_1'''')$

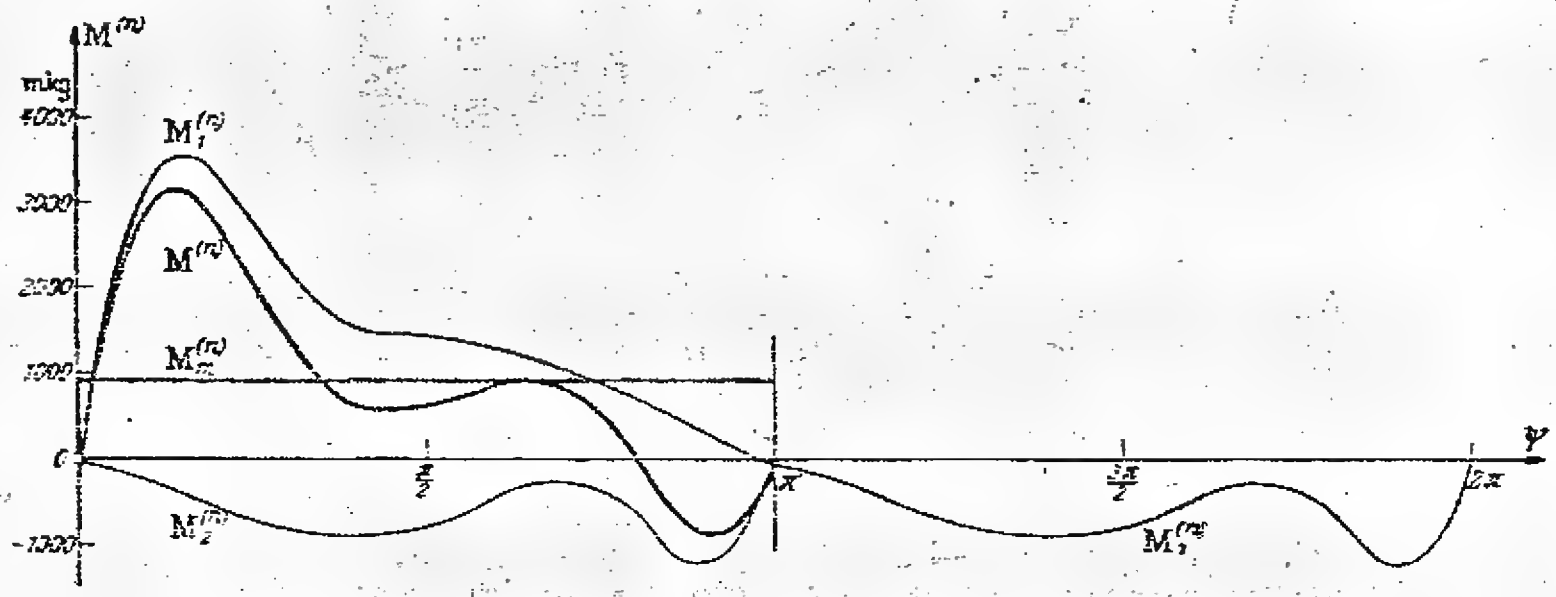


Fig. 30.—The useful torque $\mathfrak{M}^{(n)}$ and its average value $\mathfrak{M}_m^{(n)}$ for a low-speed two-cylinder two-cycle Diesel engine.

or the effective torque $\mathfrak{M}_1^{(e)} = \mathfrak{M}_1' - (\mathfrak{M}_1''' + \mathfrak{M}_1'''')$ as function of ψ. Then on the diagram we replot the same curve for each of the remaining mechanisms offset along the ψ-axis according to its crank angle, finally adding the ordinates algebraically. Thus, by superposition we obtain the resulting torque $\mathfrak{M}^{(n)}$ or $\mathfrak{M}^{(e)}$ of the entire engine. In doing this we assume that the total friction torque from the crank bearings equals the sum of the friction torques from the individual mechanisms. This assumption is certainly not exactly correct; however, since it is nearly correct, and in view of the fact that this torque is much smaller than $\mathfrak{M}'$ and $\mathfrak{M}''$, we shall use it without hesitation.

Figs. 30 to 35 show the total useful torque $\mathfrak{M}^{(n)}$ for low-speed two-stroke-cycle Diesel engines with equal crank offsets, for two, three, four, five, six, and eight cylinders. Fig. 30 also shows the original single-cylinder torque

[1] H. Schaüx, "Die Eigenschaften der Fuenfzylinder-Reihenverbrennungsmaschine", *Motorwagen* 31 (1928), p. 663; H. Schaüx, "Die Zuendfolge der vielzylindrigen Verbrennungsmaschinen", Munich and Berlin, 1938.

$M_1^{(n)}$ on which the other curves are based. Naturally, in each case the curve may be limited in length to a single period, extending between $\psi = 0$ and $\psi = 2\pi/n$, where n is the cylinder number. In the first three figures (up to $n = 4$) are shown (by light lines) the curve elements that are obtained by

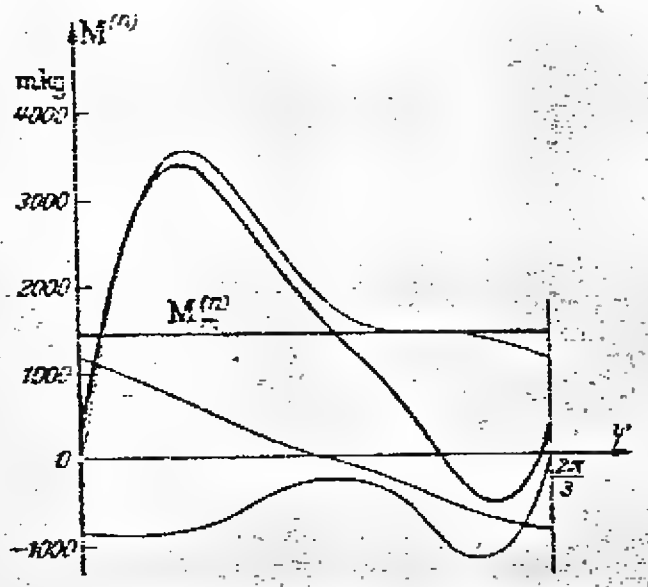


Fig. 31.—The same for a three-cylinder engine.

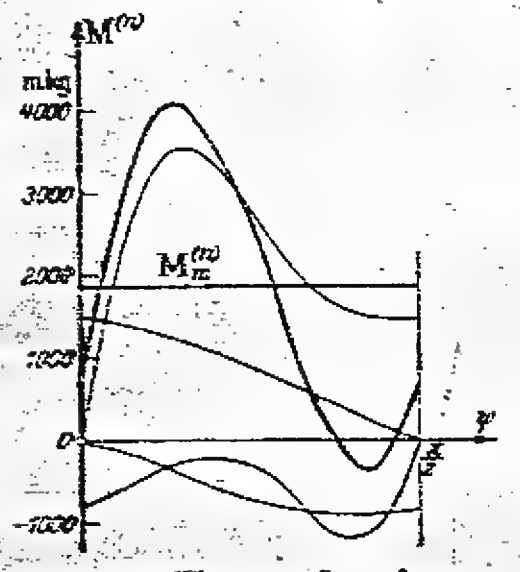


Fig. 32.—The same for a four-cylinder engine.

displacing the $M_1^{(n)}$-curve horizontally by the amounts $k \times 2\pi/n$ ($k = 0, 1, 2, \ldots n - 1$). The resultant torque $M^{(n)}$ (shown by heavy line) is obtained by addition of the above-mentioned elements and repeats itself after each period. The mean torque $M_m^{(n)}$ (7, 3) is the average ordinate of the $M^{(n)}$-curve. Exactly the same constructions for high-speed four-stroke Diesels, also with

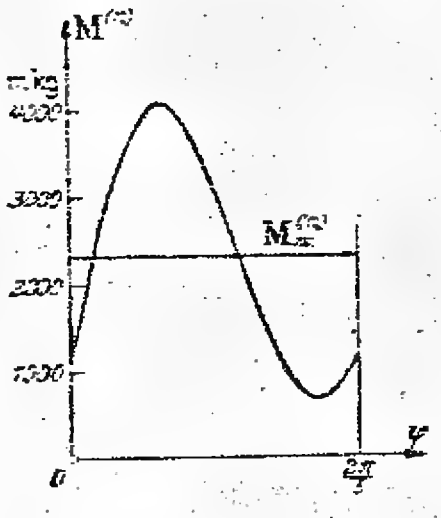


Fig. 33.—The same for a five-cylinder engine.

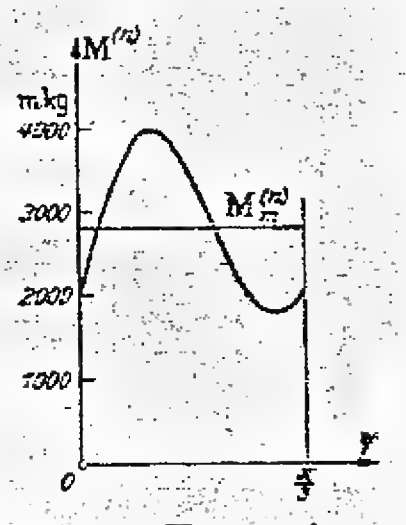


Fig. 34.—The same for a six-cylinder engine.

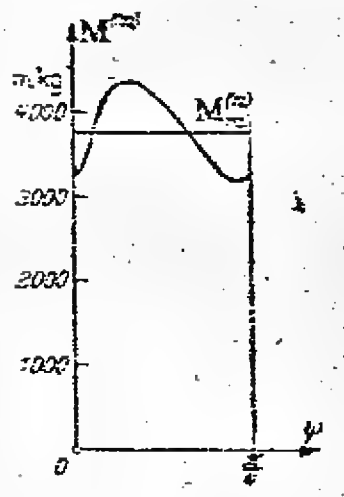


Fig. 35.—The same for an eight-cylinder engine.

constant crank offsets, are shown in Figs. 36 to 41. These are based on the $M_1^{(n)}$-curve of Fig. 19 of Sect. 8, both halves of which also appear in Fig. 36.

We may take these curves as typical. If we assume that the variation of the useful torque $M^{(n)}$ about its mean value is a measure of the quality of the power regulation then we conclude that, in general, the low-speed two-stroke engines have better power regulation than the high-speed four-

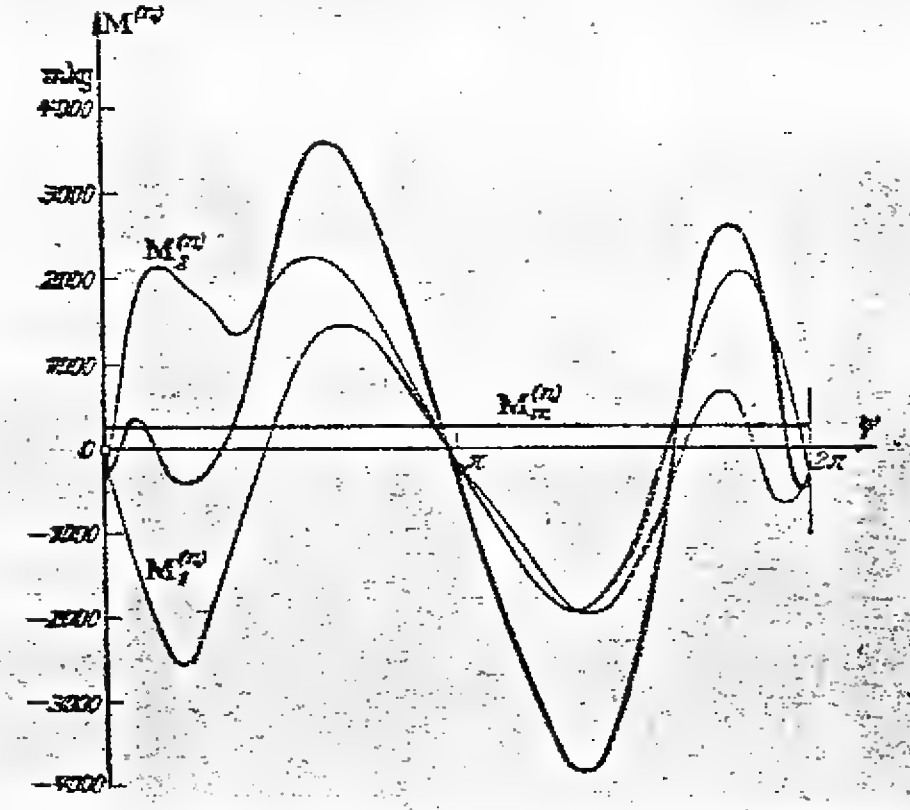


Fig. 36.—The useful torque $M^{(n)}$ and its average value $M_m^{(n)}$ for a high-speed two-cylinder four-cycle Diesel engine.

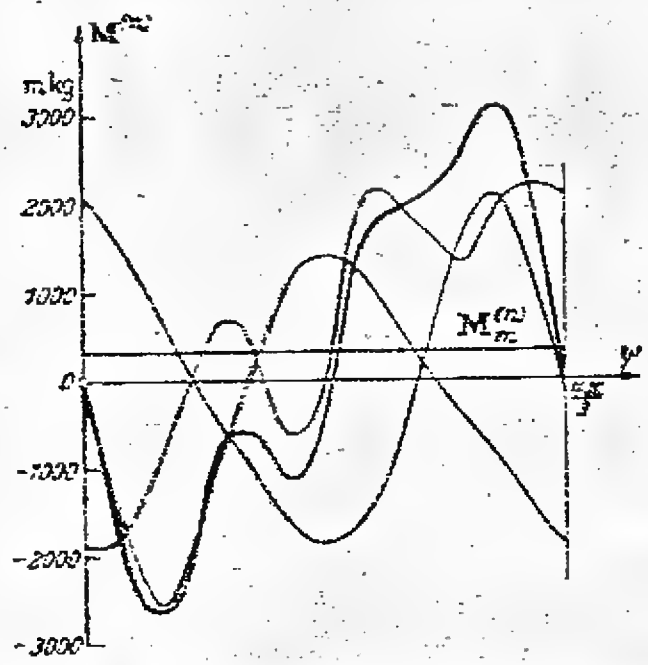


Fig. 37.—The same for a three-cylinder engine.

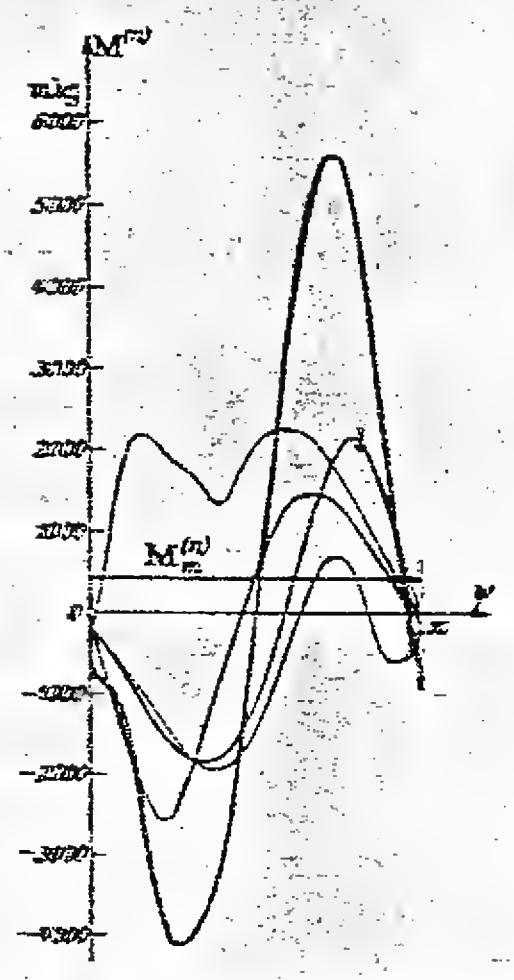


Fig. 38.—The same for a four-cylinder engine.

stroke engines, and also that power regulation improves generally with increasing number of cylinders. The four-cylinder engine is an outstanding exception; the two-stroke is less uniform than the two- and three-cylinder engines, and in the four-stroke the difference is even greater between four cylinders and two or three. Among four-stroke engines the eight-cylinder is less satisfactory than the six-cylinder in respect of power regulation. These results correspond roughly to the inertia balancing characteristics of these engines (I, 10).

That these results are typical and that all internal-combustion engines behave alike can be proved by means of the Fourier series for $\mathfrak{M}_1^{(n)}$. For example, in the four-stroke engine the Fourier series for the first component of $\mathfrak{M}_1^{(n)}$, the tangential force torque $\mathfrak{M}_1'$, is always of the type illustrated by

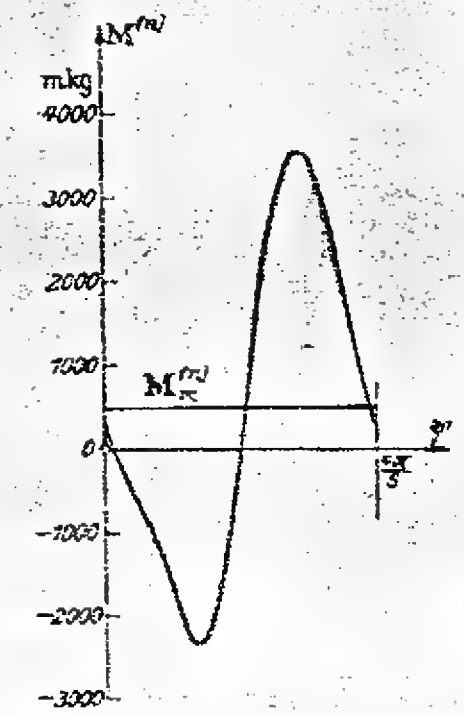


Fig. 39.—The same for a five-cylinder engine.

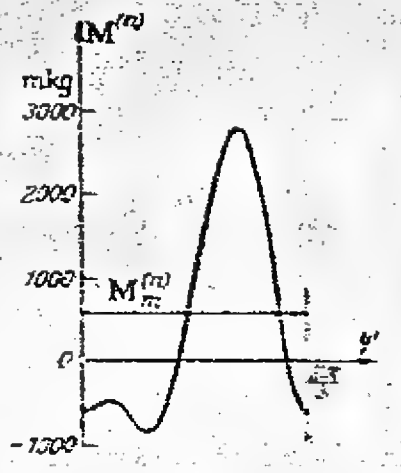


Fig. 40.—The same for a six-cylinder engine.

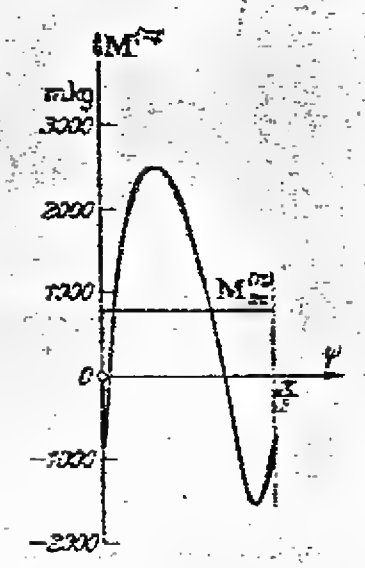


Fig. 41.—The same for an eight-cylinder engine.

the lower part of Fig. 6 of Sect. 3: c_2 and c_3 are always the most important coefficients while the succeeding coefficients become increasingly small. In the Fourier series for $\mathfrak{M}_1''$, thus essentially the series for $d\Psi/d\varphi$ (4, 7), the fourth-order term will always stand out (see Fig. 7 of Sect. 4 in which 2π is the half-period, or (4, 6a)). The remaining terms $\mathfrak{M}_1'''$ and $\mathfrak{M}_1''''$ of $\mathfrak{M}_1^{(n)}$ can play no great role, so that on the whole in high-speed engines the fourth-order term is the most important, overshadowing the terms of second and third order. In combining the Fourier series for $\mathfrak{M}_1^{(n)}$, $\mathfrak{M}_2^{(n)}$, . . . $\mathfrak{M}_z^{(n)}$ we can proceed exactly as for inertia balancing in I, 10: the Fourier series are represented by vectors $\mathfrak{e}_k^i$ which are not unit vectors but have lengths proportional to the corresponding Fourier coefficients. Thus, the first-order terms are vectors $\mathfrak{e}_1^i$ whose azimuths are those of the cranks in the crank diagram (I, 9); the second-order terms are vectors $\mathfrak{e}_2^i$ whose azimuth angles are double the corresponding crank angles, etc. Then the algebraic addition of the Fourier series $\mathfrak{M}_1^{(n)}$, $\mathfrak{M}_2^{(n)}$, . . . $\mathfrak{M}_z^{(n)}$ becomes simply the geometric addition of the vectors $\mathfrak{e}_k^i$, exactly as in I, 9 and I, 10. From this we know that with uniform crank

spacing these vector series fail to vanish only when the order j of the Fourier term equals n or an integral multiple of n. Hence, the principal term of the sum $\mathbf{M}^{(n)}$ for the n-cylinder engine is of the order n, and is combined only with terms of order $2n$, $3n$, etc. But since the fourth-order term is always the most important in $\mathbf{M}_1''$ for high-speed engines, it follows that a high-speed four-cylinder engine will show especially large variations in $\mathbf{M}^{(n)}$ and particularly poor power regulation.

Because it was shown in I, 12c that the six-cylinder engine with crank angles alternately 90° and 30° shows definite superiority in respect to inertia balance compared to the engine with equal crank angles, we present in Figs. 42 and 43 the $\mathbf{M}^{(n)}$-diagrams (two- and four-cycle) for this engine for the

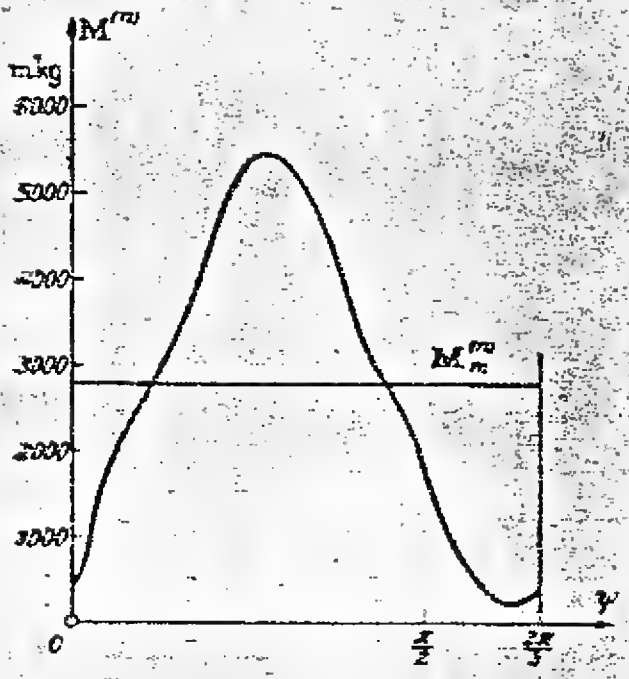


Fig. 42.—The useful torque $\mathbf{M}^{(n)}$ and its average value $\mathbf{M}_m^{(n)}$ for a low-speed six-cylinder two-cycle Diesel engine with crank offsets of 90° and 30°.

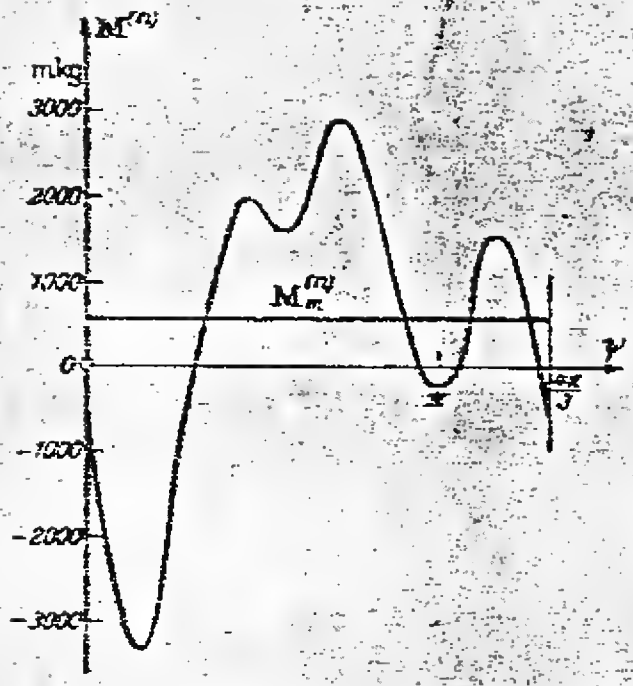


Fig. 43.—The same for a high-speed six-cylinder four-cycle Diesel engine with crank offsets of 90° and 30°.

same single-cylinder $\mathbf{M}_1^{(n)}$ as in Figs. 34 and 40. The vector diagram indicates that in the Fourier series for $\mathbf{M}^{(n)}$ only terms of order 3, 9, 15, ... enter. If we compare Fig. 42 with Fig. 34, and Fig. 43 with Fig. 40 (in which the period is now twice as great as before, so that Figs. 34 and 40 must be imagined repeated again) we discover that particularly in the two-stroke engine the unequal crank angles are definitely inferior to equal crank angles with respect to power regulation; improved inertia balance is obtained here at the cost of inferior power regulation.

As soon as the $\mathbf{M}^{(n)}$-curve for the whole engine is found we can calculate finally the required flywheel characteristics and the operation of the engine, using the first method (Sect. 8 and 9), at least with the approximation that makes use only of the torque surplus and deficit $\mathbf{U} = \mathbf{M}^{(n)} - \mathbf{L}$. In the flywheel calculation for the in-line engine there is the advantage that for more

than three cylinders only two areas F_i, F_{i+1} (8, 6) need to be planimetered in order to determine the moment of inertia A. For more precise calculations, however, we also need the Ψ-curve of the whole engine. Generally, we are satisfied with the approximation (8, 13) and (9, 8) and need only determine the average value Ψ_{m1} for the first mechanism, from which is found $\Psi_m = n\Psi_{m1}$ for the n-cylinder engine.

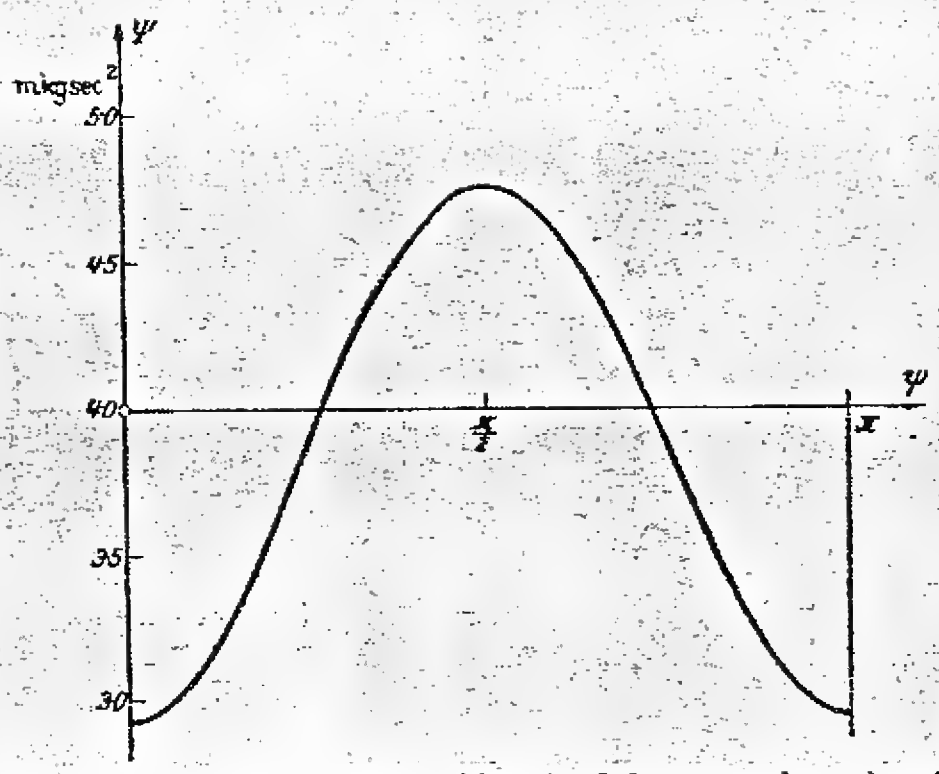


Fig. 44.—Equivalent moment of inertia of the two-cycle engine of Fig. 1, Part I, with two cylinders.

If we wish to make a calculation with the still more exact formulæ (8, 14) or (8, 15), (8, 16) and (9, 9), in addition to the $\mathfrak{M}^{(n)}$-curve for the whole engine we must also construct its Ψ-curve by the same method of superposition. The results for two-stroke engines are shown in Figs. 44–49, for four-stroke in Figs. 50–55, both with equal crank angles, for the

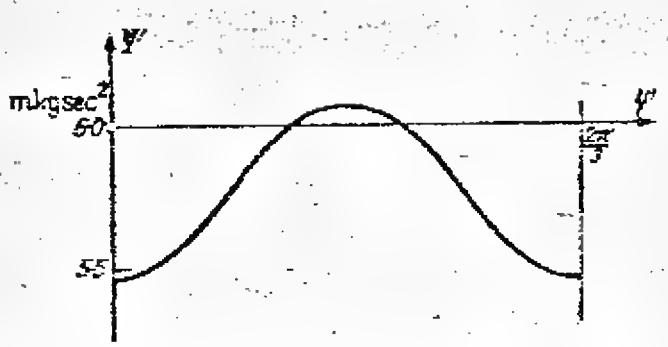


Fig. 45.—The same with three cylinders.

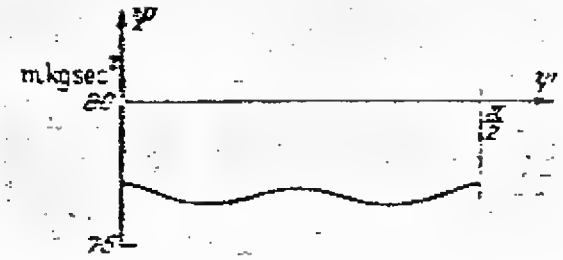


Fig. 46.—The same with four cylinders.

mechanism of Fig. 1 of Part I whose single-cylinder Ψ-curve appears in Fig. 7 of Sect. 4. Since these curves are quite typical we may draw the following general conclusions from them: In two-stroke engines both the absolute and the relative variations in Ψ diminish with increasing number of cylinders and are barely noticeable with four or more cylinders. In four-

stroke engines this rule is badly violated by four and six cylinders, as is clearly shown by the table on p. 131, which is based on Fig. 7 and Figs.

Fig. 47.—The same with five cylinders.

Fig. 48.—The same with six cylinders.

Fig. 49.—The same with eight cylinders.

44–55. This is simply the result of the fact that in the one-cylinder engine (Fig. 7), in the Fourier development of the equivalent moment of inertia Ψ

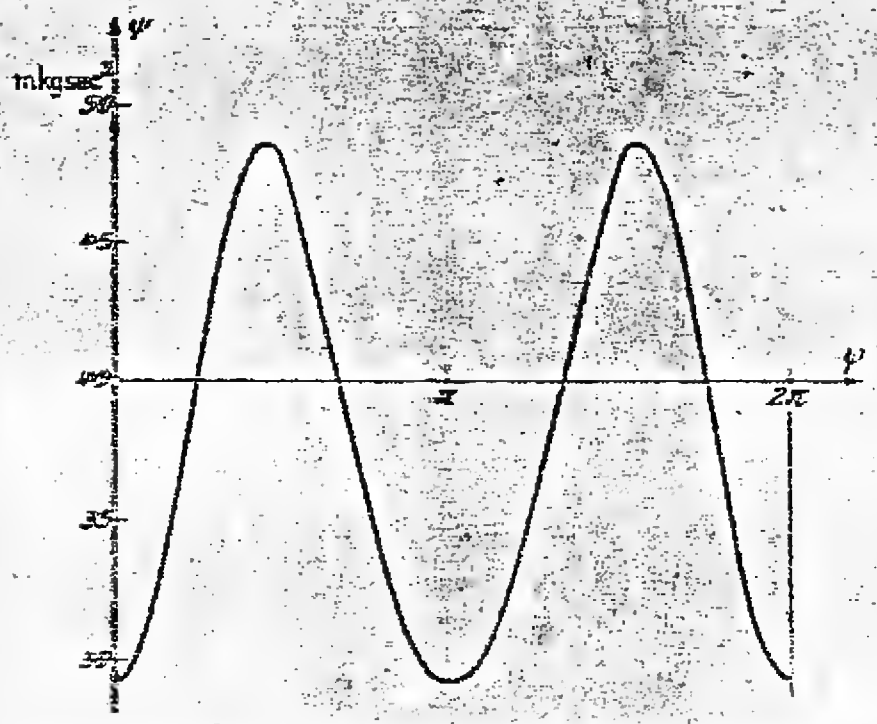


Fig. 50.—Equivalent moment of inertia of the four-cycle engine of Fig. 1, Part I, with two cylinders.

for a period 2π, the terms of second and (to less extent) third order are by far the most important. Because the cranks are arranged in symmetrical

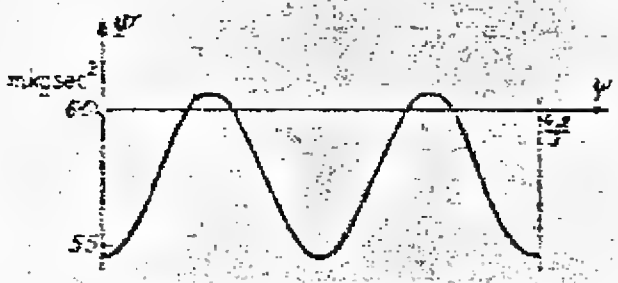


Fig. 51.—The same with three cylinders.

pairs in the four-stroke engine with an even number of cylinders $2n$, the relative variation in Ψ is the same in this engine as in a two-stroke engine with n cylinders.

Fig. 52.—The same with four cylinders.

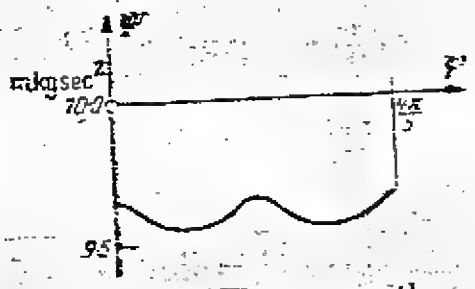


Fig. 53.—The same with five cylinders.

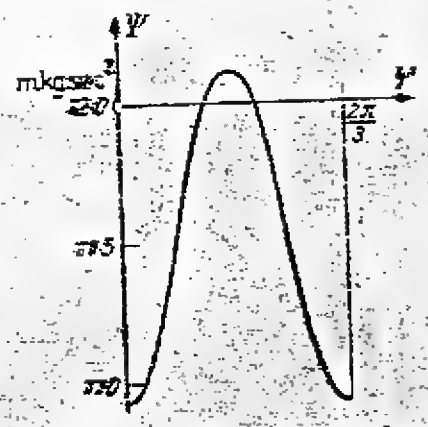


Fig. 54.—The same with six cylinders.

Fig. 55.—The same with eight cylinders.

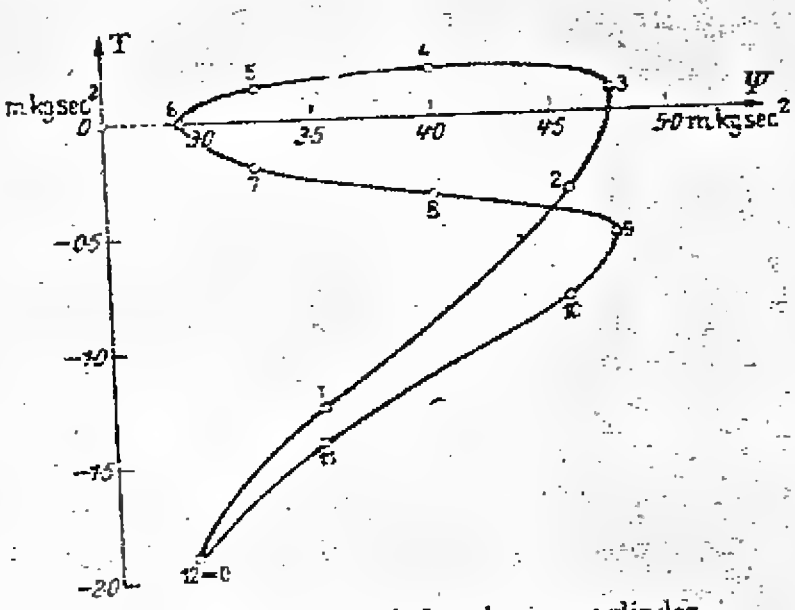


Fig. 56.—Ψ-T graph for the two-cylinder four-cycle engine.

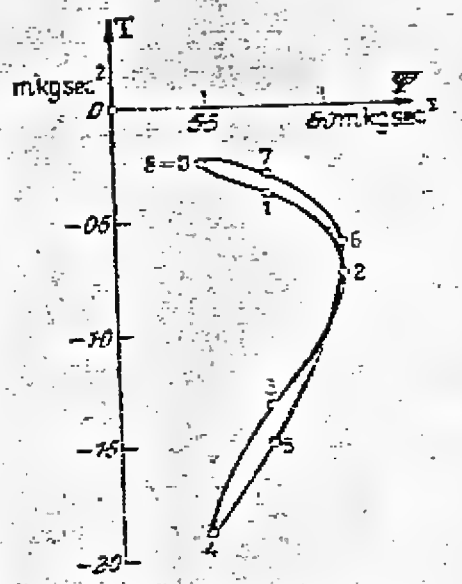


Fig. 57.—The same for the three-cylinder engine.

If it is desired to calculate the flywheel characteristics and the operation of the in-line engine by the second method (Sect. 10 and 11), in addition to the Ψ-curve, the T-curve (10, 4) must be constructed by superposition of

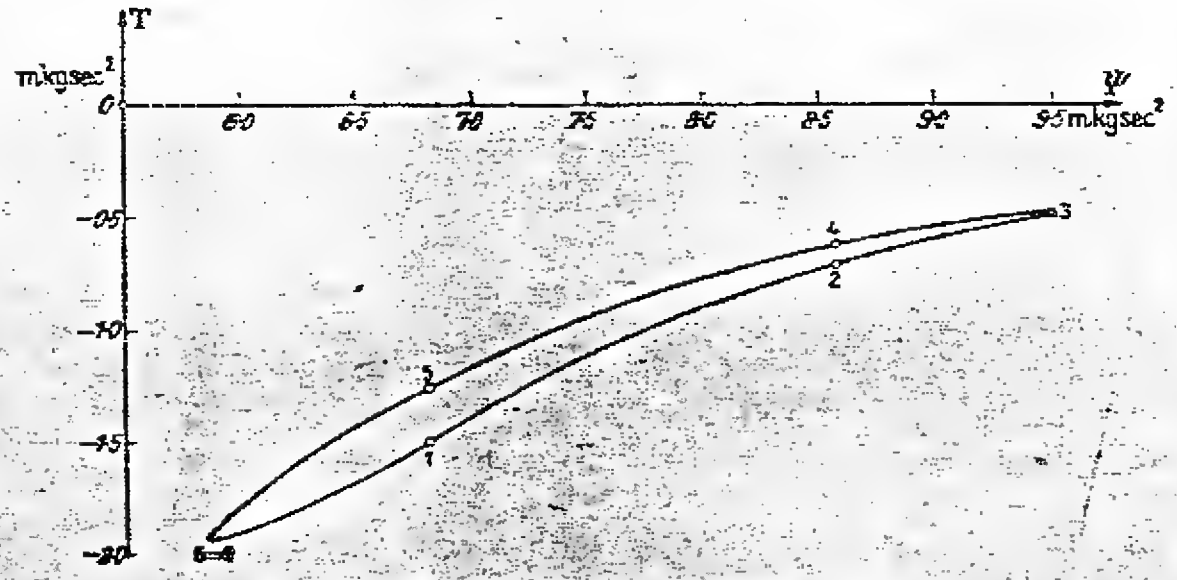


Fig. 58.—The same for the four-cylinder engine.

elements of the T-curve for the one-cylinder engine. Next, the Ψ-T-graph is plotted. Figs. 56–61 show typical Ψ-T-graphs for two- to eight-cylinder four-stroke engines for the same T-curve (Fig. 22) that was the basis of the Ψ-T-graph shown in Fig. 23 of Sect. 10 for the one-cylinder engine. To

Fig. 59.—The same for the five-cylinder engine.

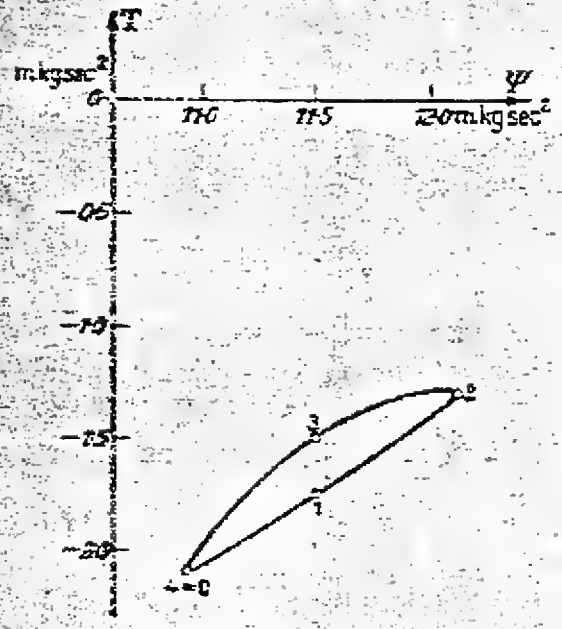


Fig. 60—The same for the six-cylinder engine.

save space in these figures the T-axis has been shown offset to the right from its true position. The numbered points correspond to crank rotation intervals $\psi = \pi/6$. The diagrams repeat themselves every $24/n$ numbers since equal crank offsets are assumed (for $n = 5$ this gives a non-integer). According to (11, 3) the required moment of inertia A of the flywheel is proportional to the length $R'R''$ cut from the T-axis by the lowest and

Relative Variation of Ψ.

No. Cylinders	2-Stroke %	4-Stroke %
1	23	23
2	23	23
3	5	5
4	0·5	23
5	0·5	0·5
6	0·3	5
8	0·3	0·5

highest 45°-tangents to the curve. From the table below it appears that the two-cylinder four-stroke engine is approximately equal to the one-

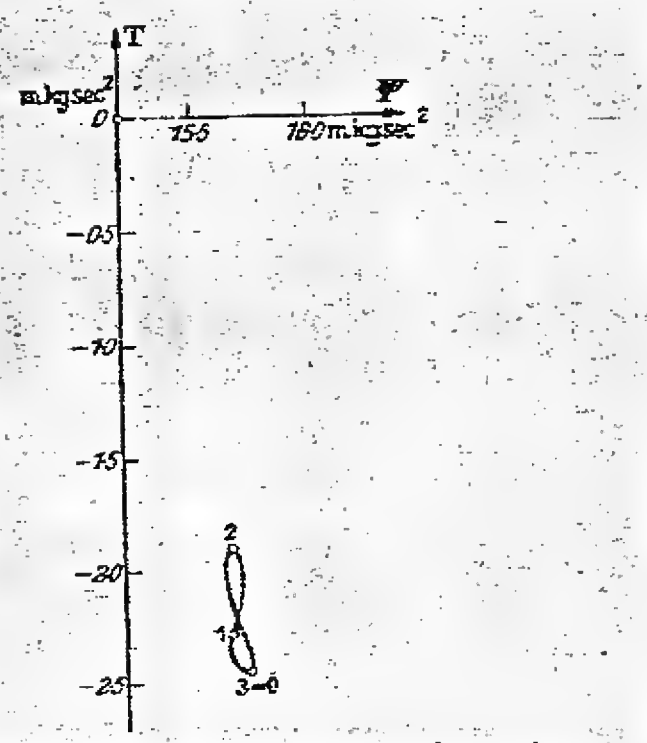


Fig. 61.—The same for the eight-cylinder engine.

cylinder four-stroke engine, and that again the four-cylinder engine is relatively poor, while the eight-cylinder is worse than the six-cylinder. Thus, the conclusions drawn from the $M^{(n)}$-curve are substantiated.

No. Cylinders	$R' R''$ m.kg.sec.²
1	2·45
2	2·50
3	1·67
4	2·26
5	1·36
6	0·48
8	0·62

14. V- and Fan-like Engines with centrally linked Connecting-rods.—The transition from the in-line engine to the V-engine (Fig. 34 of I, 16), the fan-like engine (Fig. 40 of I, 18), and to the X-engine (Fig. 44 of I, 20), with one or more cylinders per row, offers no difficulty so long as the individual mechanisms are identical and all connecting-rods are attached directly to the crankpin or are centrally linked.

First, the tangential force torque $\mathfrak{M}'$ for each V or bank of cylinders can be found by superposition based on the torque curve for one cylinder $\mathfrak{M}_1'$, as was done for the in-line engine. It makes no difference in $\mathfrak{M}'$ whether two pistons with parallel lines of stroke work on two cranks that are offset by an angle δ (as in the two-cylinder in-line engine), or two pistons with lines of stroke making the angle δ act on the same crank (as in the V-engine). It is only essential to have equal firing intervals. We use the symbols of Fig. 34 of I, 16, measuring the crank rotation from its symmetry position. Consequently, in order to obtain the tangential force torque $\mathfrak{M}'$ of a single V we must offset the $\mathfrak{M}'$-curve for one cylinder the distance $+\delta/2$ along the φ-axis and repeat with an offset $-\delta/2$, finally adding the ordinates algebraically. In the three-cylinder bank (W-type engine) the original $\mathfrak{M}'$-curve is not offset and is combined with two identical curves that are displaced along the φ-axis by $\pm\delta$. The X-engine is treated correspondingly.

Likewise, the Ψ-curves are superimposed, provided the crank weight has been divided among the individual mechanisms. Alternatively, the crank weight can be omitted in finding the Ψ-curve for a single mechanism and only after the superposition of the curves is it included in the form k^2G/g.

In getting the torque $\mathfrak{M}'''$ corresponding to the potential energy, the superposition is carried out as follows: for the V-engine, at first assumed to be upright, on comparing Fig. 34 of Part I with Fig. 8 of Sect. 5, we find it necessary to write equations (5, 2) once with $\varphi + \delta/2$ instead of φ and $\delta/2$ instead of δ, and again with $\varphi - \delta/2$ instead of φ and $-\delta/2$ instead of δ, finally adding both expressions. We then get

$$\mathfrak{M}''' = -\Big\{\big[Q^*(1+\cos\delta) - 2Q^{*\prime\prime}\big]\sin\varphi + 2Q^*\cos\frac{\delta}{2}\,(B_2\cos\delta\sin 2\varphi - \\ - B_4\cos 2\delta\sin 4\varphi + -\cdots)\Big\}, \tag{1}$$

where

$$Q^* = r\left(\frac{s'}{l}\,G' + G''\right), \qquad Q^{*\prime\prime} = \frac{1}{2}\,sG + r\left(1 - \frac{s'}{l}\right)G'. \tag{2}$$

By introducing X^*, the steady component of the longitudinal force (I, 16, 5), we get this in the simple form:

$$\mathfrak{M}'' = -\frac{g}{\omega^2}\int_0^\varphi X^*\,d\varphi, \tag{3}$$

which can also be obtained directly from (5, 9). For the vertical three-row (W-type) engine the expression (5, 2) must be used three times, once with $\psi + \delta$ instead of ψ and with $+\delta$, once with $\psi - \delta$ instead of ψ and with $-\delta$ instead of δ, and finally with ψ and with $\delta = 0$. Addition gives

$$M''' = -\frac{g}{\omega^2}\int_0^{\psi} X^* \, d\psi, \tag{4}$$

in which X^* is the steady part of the longitudinal force (I, 18, 3). The same formula also holds for the X-type engine, the force being given by (I, 20, 1).

For the internal forces of the mechanism we take equations (6, 15)–(6, 19) and write them with the same changes in ψ and δ as above. Here, we must remember that the piston force K for each piston is a function of the crank rotation ψ (given by the graphical construction of Fig. 2 in Sect. 3). Therefore, the several K's will differ in phase and must be distinguished by subscripts designating the phase differences; in the V-engine, $K_{\psi-\delta/2}$, $K_{\psi-\delta/2}$; in the W-engine K_{ψ}, $K_{\psi+\delta}$, $K_{\psi-\delta}$. The formulæ (6, 15)–(6, 17) will then give the gudgeon-pin forces, the cylinder side-wall forces and the connecting-rod forces in each mechanism. Formulæ (6, 18) must be added within each V- or W-element, giving for the upright V-engine the total crankpin forces:

$$\left.\begin{aligned}
P_1 = {} & K_{\psi-\delta/2}\left[\sin\left(\psi+\frac{\delta}{2}\right)+\frac{1}{2}\lambda\sin(2\psi+\delta)\right]+K_{\psi-\delta/2}\left[\sin\left(\psi-\frac{\delta}{2}\right)+\frac{1}{2}\lambda\sin(2\psi-\delta)\right]+\\
& +2\left(1-\frac{s'}{l}\right)G'\sin\psi-\left(\frac{s'}{l}G'+G''\right)\left\{\frac{r\omega^2}{g}\left[\cos\delta\sin 2\psi-\frac{1}{2}\lambda\left(\cos\frac{\delta}{2}\sin\psi-\right.\right.\right.\\
& \left.\left.\left.-3\cos\frac{3\delta}{2}\sin 3\psi\right)\right]-\cos\frac{\delta}{2}\left(2\cos\frac{\delta}{2}\sin\psi+\lambda\cos\delta\sin 2\psi\right)\right\},\\
P_2 = {} & K_{\psi-\delta/2}\left[\cos\left(\psi+\frac{\delta}{2}\right)-\frac{1}{2}\lambda[1-\cos(2\psi+\delta)]\right]+K_{\psi-\delta/2}\left[\cos\left(\psi-\frac{\delta}{2}\right)-\right.\\
& \left.-\frac{1}{2}\lambda[1-\cos(2\psi-\delta)]\right]-2\left(1-\frac{s'}{l}\right)G'\left(\frac{r\omega^2}{g}-\cos\psi\right)-\\
& -\left(\frac{s'}{l}G'+G''\right)\left\{\frac{r\omega^2}{g}\left[1+\cos\delta\cos 2\psi+\frac{1}{2}\lambda\left(\cos\frac{\delta}{2}\cos\psi+3\cos\frac{3\delta}{2}\cos 3\omega\right)\right]-\right.\\
& \left.-\cos\frac{\delta}{2}\left[2\cos\frac{\delta}{2}\cos\psi-\lambda(1-\cos\delta\cos 2\psi)\right]\right\}.
\end{aligned}\right\} \tag{5}$$

The terms containing K can be evaluated graphically from (6, 18a). This need be done for only one cylinder and the result then repeated with displacement $+\delta/2$ and again with displacement $-\delta/2$, and finally added.

It must be emphasized that expressions (5) give only total forces on a crankpin. If we were to calculate the friction torque from this force we would find that the torque was considerably underestimated. The reason is that each connecting-rod is individually attached to the crank, so that its friction torque must naturally depend on the bearing force (6, 18) exerted by it. The sum of the separate torques gives the total friction torque on the

crankpin. These remarks apply equally to the fan-like or W-engine and to the radial engine with centrally attached connecting-rods (discussed in Sect. 15 following).

For the upright W-type engine we have

$$\left.\begin{aligned}
P_1 &= K_\psi\left(\sin\psi+\frac{1}{2}\lambda\sin 2\psi\right)+K_{\psi+\delta}\left[\sin(\psi+\delta)+\frac{1}{2}\lambda\sin(2\psi+2\delta)\right]+\\
&\quad+K_{\psi-\delta}\left[\sin(\psi-\delta)+\frac{1}{2}\lambda\sin(2\psi-2\delta)\right]+3\left(1-\frac{s'}{l}\right)G'\sin\psi-\\
&\quad-\left(\frac{s'}{l}G'+G''\right)\left\{\frac{1}{2}\frac{r\omega^2}{g}\left\{(1+2\cos 2\delta)\sin 2\psi-\frac{1}{2}\lambda\left[(1+2\cos\delta)\sin\psi-\right.\right.\right.\\
&\quad\left.\left.\left.-3(1+2\cos 3\delta)\sin 3\psi\right]\right\}-(2+\cos 2\delta)\sin\psi-\frac{1}{2}\lambda(1+\cos\delta+\cos 3\delta)\sin 2\psi\right\},\\
P_2 &= K_\psi\left[\cos\psi-\frac{1}{2}\lambda(1-\cos 2\psi)\right]+K_{\psi+\delta}\left\{\cos(\psi+\delta)-\frac{1}{2}\lambda\left[1-\cos(2\psi+2\delta)\right]\right\}+\\
&\quad+K_{\psi-\delta}\left\{\cos(\psi-\delta)-\frac{1}{2}\lambda\left[1-\cos(2\psi-2\delta)\right]\right\}-3\left(1-\frac{s'}{l}\right)G'\left(\frac{r\omega^2}{g}-\cos\psi\right)-\left(\frac{s'}{l}G'+G''\right)\cdot\\
&\quad\cdot\left\{\frac{1}{2}\frac{r\omega^2}{g}\left\{3+(1+2\cos 2\delta)\cos 2\psi-\frac{1}{2}\lambda\left[(1+2\cos\delta)\cos\psi+3(1+2\cos 3\delta)\cos 3\psi\right]\right\}-\right.\\
&\quad\left.-(2+\cos 2\delta)\cos\psi+\frac{1}{2}\lambda\left[(1+2\cos\delta)-(1+\cos\delta+\cos 3\delta)\cos 2\psi\right]\right\}.
\end{aligned}\right\}\tag{6}$$

Corresponding relations are found for the X-type engine.

Finally, for all three types of engines, from (6, 19) the forces on the main bearings are found to be

$$\left.\begin{aligned}
P_3 &= P_1+G\sin\psi,\\
P_4 &= P_2-G\left(\frac{s\omega^2}{g}-\cos\psi\right).
\end{aligned}\right\}\tag{7}$$

For the inverted V- and W-engines it is only necessary to change the algebraic signs of the weight symbols Q^*, $Q^{*\prime\prime}$, $Q^{*\prime\prime\prime}$ or Q, Q'', Q''' and G, G', G'', and then to measure the angle ψ from the lowest rather than the highest crank position.

Thus we have obtained all basic information from which the $M^{(a)}$-curve, the Ψ-curve and the T-curve for a single crank of a V-, W- or X-type engine can be developed.

The curves so obtained for a single crank can be extended to multi-bank V-, W- and X-type engines exactly as the curves for a single cylinder were extended to the in-line engine (Sect. 13). However, in many cases it will be sufficiently exact if in a V-engine we ignore the difference in potential energies of the two mechanisms attached to the same crank, and furthermore make the (approximately correct) assumption that the total friction torque of each V equals the sum of the individual torques of the two mechanisms that belong to it. With these assumptions, we can superimpose

the curves for all the individual cylinders (assumed now to be identical), offsetting them with respect to one another according to the actual firing intervals. An exactly similar procedure would be followed with W- or X-type engines. For example, if we consider a six-cylinder four-stroke V-type engine whose cranks are offset by 120° while the cylinder rows are at right angles to each other (I, 16), the best firing order is 12, 11, 22, 21, 32, 31 (the first figure of each pair denotes the crank, the second the cylinder-bank as in Fig. 34 of Part I). As can be easily shown, this gives firing intervals of alternately 90° and 150°. Consequently, in making the superposition mentioned above, the single-cylinder curve must be drawn six times, beginning at $\psi = 0°$, 90°, 240°, 330°, 480° and 570°. This is exactly the same procedure that was followed with the normal six-cylinder four-stroke in-line engine having crank-offsets alternately 90° and 30°; consequently, the $\mathfrak{M}^{(a)}$-curve of the six-cylinder V-engine that is being considered here coincides with Fig. 43 of Sect. 13. If we ignore small differences in potential energies and in the friction relationships the two engines will behave exactly alike in regard to power regulation.

15. The Radial Engine with all Connecting-rods centrally linked.—The superposition described in Sect. 14 naturally is also possible for the radial engine in which all connecting-rods are attached directly to the crankpin (Fig. 42 of I, 19). Again, we begin by ignoring the differences in the potential energies of the individual mechanisms that make up the engine, as well as the difference between the actual friction torque $\mathfrak{M}''''$ and the sum of the friction torques of the several mechanisms. We then find no difference between the power regulation of the symmetrical radial engine and that of the normal in-line engine with the same number of cylinders and equal crank-offsets. Consequently, all of the Figs. 30–41 and 44–61 of Sect. 13 can be applied also to radial engines with centrally attached connecting-rods, as can the conclusions given in the same place.

For a more exact investigation of radial engines it is advantageous to obtain the torques $\mathfrak{M}'''$ (potential energy) and $\mathfrak{M}''''$ (friction) more carefully. Then, after the $\mathfrak{M}'$-curve and the Ψ-curve of the whole engine are found by superposition (exactly as for the usual in-line engine) the torque corresponding to the potential energy $\mathfrak{M}'''$ is found in the same way as for the V- and W-type engines. For example, for the horizontal two-cylinder radial engine (Fig. 36 of I, 17) we find

$$\mathfrak{M}''' = -\frac{g}{\omega^2}\int_0^{\psi} X^*\, d\psi, \tag{1}$$

where X^* is the steady part of the force X (I, 17, 1). For the upright two-cylinder radial engine

$$\mathfrak{M}''' = -\frac{g}{\omega^2}\int_{\pi/2}^{\psi} Y^*\, d\psi, \tag{2}$$

where Y^* is the steady part of the force Y (I, 17, 1). For the symmetrical radial engine with one line of stroke directed vertically upward

$$M''' = -\frac{g}{\omega^2}\int_0^\psi X^*\,d\psi, \tag{3}$$

where X^* represents the steady component of the force X. (In (I, 19, 4) and (I, 19, 5) X^* is the symbol used for the whole first-order force containing both steady and unsteady components.) For radial engines with many cylinders this torque is normally very small.

When we consider the cyclic symmetry of the mechanism and use equations (I, 19, 2) and (I, 19, 3) we get from (6, 18) and (6, 18a) the following simple expressions for the components of the crankpin force: In the horizontal two-cylinder radial engine

$$\left.\begin{aligned} P_1 &= OF_{\psi+\pi/2} + OF_{\psi-\pi/2} + 2\left(1-\frac{\varepsilon'}{l}\right)G'\sin\psi + \left(\frac{\varepsilon'}{l}G' + G''\right)\frac{r\omega^2}{g}\sin 2\psi,\\ P_2 &= EL_{\psi+\pi/2} + EL_{\psi-\pi/2} - 2\left(1-\frac{\varepsilon'}{l}\right)G'\left(\frac{r\omega^2}{g} - \cos\psi\right) -\\ &\qquad - \left(\frac{\varepsilon'}{l}G' + G''\right)\frac{r\omega^2}{g}(1-\cos 2\psi). \end{aligned}\right\} \tag{4}$$

In the vertical two-cylinder radial engine

$$\left.\begin{aligned} P_1 &= OF_\psi + OF_{\psi-\pi} + 2(G'+G'')\sin\psi - \left(\frac{\varepsilon'}{l}G' + G''\right)\frac{r\omega^2}{g}\sin 2\psi,\\ P_2 &= EL_\psi + EL_{\psi-\pi} + 2(G'+G'')\cos\psi - \left[2\left(1-\frac{\varepsilon'}{l}\right)G' +\right.\\ &\qquad \left. + \left(\frac{\varepsilon'}{l}G' + G''\right)(1+\cos 2\psi)\right]\frac{r\omega^2}{g}. \end{aligned}\right\} \tag{5}$$

In the symmetrical three-cylinder radial engine with one line of stroke vertically upward

$$\left.\begin{aligned} P_1 &= OF_\psi + OF_{\psi-2\pi/3} + OF_{\psi-4\pi/3} - 3\left(1-\frac{\varepsilon'}{l}\right)G'\sin\psi -\\ &\qquad - \frac{3}{2}\left(\frac{\varepsilon'}{l}G' + G''\right)\left(\frac{3}{2}\frac{r\omega^2}{g}\lambda\sin 3\psi - \sin\psi - \frac{1}{2}\lambda\sin 2\psi\right),\\ P_2 &= EL_\psi + EL_{\psi-2\pi/3} + EL_{\psi-4\pi/3} - 3\left(1-\frac{\varepsilon'}{l}\right)G'\left(\frac{r\omega^2}{g} - \cos\psi\right) -\\ &\qquad - \frac{3}{2}\left(\frac{\varepsilon'}{l}G' + G''\right)\left[\frac{r\omega^2}{g}\left(1+\frac{3}{2}\lambda\cos 3\psi\right) - \cos\psi - \frac{1}{2}\lambda\cos 2\psi\right]. \end{aligned}\right\} \tag{6}$$

In general, for symmetrical p-cylinder radial engines with one line of stroke vertically upward,

$$\left.\begin{aligned} P_1 &= \sum_{i=1}^{p} OF_{\psi+\delta_i} + p\left[\left(1-\frac{1}{2}\frac{s}{l}\right)G' + \frac{1}{2}G''\right]\sin\psi, \\ P_2 &= \sum_{i=1}^{p} EL_{\psi+\delta_i} - p\left[\left(1-\frac{1}{2}\frac{s}{l}\right)G' + \frac{1}{2}G''\right]\left(\frac{r\omega^2}{g} - \cos\psi\right) \end{aligned}\right\} \quad (p>3), \quad \delta_i = (i-1)\,\delta. \qquad (7)$$

The components of the main bearing forces are again given by the general formulæ

$$\left.\begin{aligned} P_3 &= P_1 + G\sin\psi, \\ P_4 &= P_2 - G\left(\frac{s\omega^2}{g} - \cos\psi\right). \end{aligned}\right\} \qquad (8)$$

In radial engines with an odd number of cylinders and with one line of stroke directed vertically downward it is necessary to change the algebraic signs of all weight terms and to measure ψ from the lowest crank position in using equations (3), (6), (7) and (8).

In this manner we have obtained all basic information needed for calculating the power regulation of symmetric radial engines. In multi-bank radial engines we proceed from the solution for a single bank to that of the whole engine, exactly as the in-line engine was treated by generalizing from the single-cylinder solution. Less precisely, but much more simply, we can also proceed from the solution for a single cylinder (to which only the pth part of the crank is assigned), finding the $\mathfrak{M}^{(e)}$-curve, the Ψ-curve and the T-curve of the whole engine simply by superposition of single-cylinder curves with offsets according to the firing intervals.

16. V-, W-, and Radial Engines with Articulated Connecting-rods.—We have just treated the case where all the connecting-rods that belong to a crank are attached directly to the crankpin. Next, we consider the more usual situation (as in I, 5) in which for each crank there is a master connecting-rod to which the other rods (articulated) are attached individually (Figs. 46 and 47, I, 21). Fig. 62 shows the master rod and one articulated rod, the ith, schematically, as well as the symbols to be used in the analysis, which are the same as before. The subscript 0 refers to the master mechanism, i to the ith mechanism. The angle δ_i between the axes of the two cylinders normally equals the offset angle δ_i', but this is not necessarily the case here. It is assumed again that the centres of gravity S_0 and S_i lie on the axes of the rods A_0B_0 and A_iB_i.

In Fig. 62 M_0 is the instantaneous centre for the master rod; from it the instantaneous centre of the ith rod is found as the intersection of A_iM_0 with a perpendicular to the ith line of stroke OB_i at B_i. We erect at O perpendiculars to both lines of stroke, determine the intersection D_0 of A_0B_0 with the first of these perpendiculars, and draw D_0H_i parallel to B_0A_i, where H_i lies on A_0A_i or its extension. Then the triangles $B_0A_iA_0$ and $D_0H_iA_0$ are similar, likewise the triangles $B_0A_0M_0$ and D_0A_0O; consequently

OH_i is parallel to M_0A_i. Finally, we draw H_iD_i parallel to A_iB_i so that the triangle $B_iA_iM_i$ is similar to D_iH_iO. From this construction, relations for the velocities v_0, v_i, w_i and $r\omega$ of points B_0, B_i, A_i and A_0 are found:

$$\frac{v_0}{r\omega} = \frac{M_0B_0}{M_0A_0} = \frac{r_0'}{r}, \qquad \frac{v_i}{w_i} = \frac{M_iB_i}{M_iA_i} = \frac{r_i'}{r_i}, \qquad \frac{w_i}{r\omega} = \frac{M_0A_i}{M_0A_0} = \frac{r_i}{r} \tag{1}$$

and therefore

$$\frac{v_i}{r\omega} = \frac{v_i}{w_i}\cdot\frac{w_i}{r\omega} = \frac{r_i'}{r}. \tag{2}$$

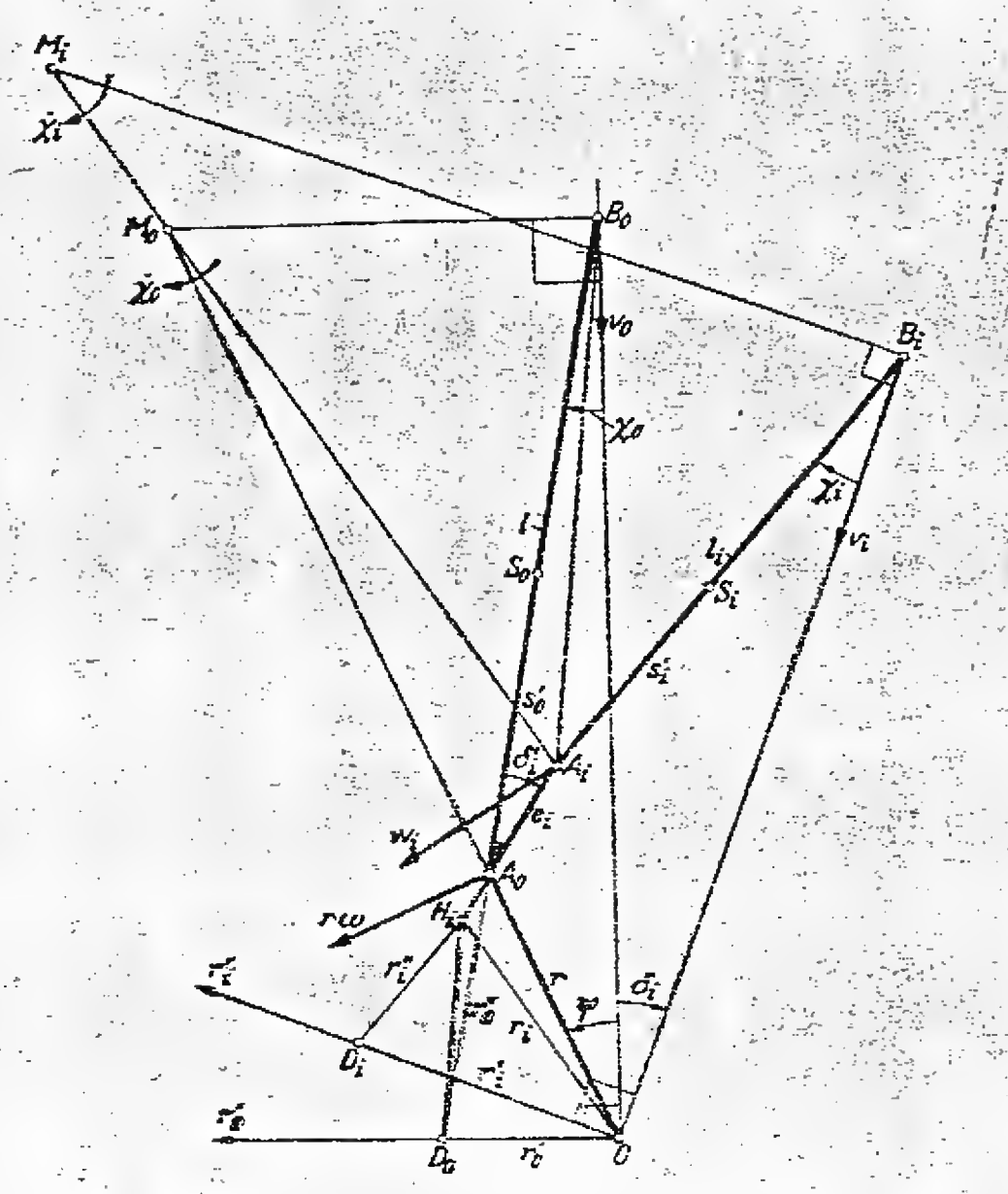


Fig. 62.—The master connecting-rod and the ith articulated rod.

Consequently, from the piston forces K_0 and K_i for the master piston and the ith piston we get the corresponding tangential-force torques

$$\mathfrak{M}_0' = K_0\frac{v_0}{\omega} = K_0r_0', \qquad \mathfrak{M}_i' = K_i\frac{v_i}{\omega} = K_ir_i', \tag{3}$$

or the tangential forces

$$P_0 = K_0\frac{r_0'}{r}, \qquad P_i = K_i\frac{r_i'}{r}. \tag{4}$$

The graphical determination of M_0' or P_0 for the master mechanism has been given in Fig. 2 of Sect. 3. Fig. 63 shows the calculation of $\mathfrak{M}_i'$ or P_i. In order to locate the two dead-centres B_{i0} and B_{i0}' of the ith mechanism we construct the crank circle f_0 and determine the geometrical path f_i of the bearing point of the ith connecting-rod on the master rod. The two dead-centres are the points lying on the ith line of stroke from which circular arcs of radius l_i can be drawn tangent to the nearer and farther sides of f_i. These two tangent arcs are extended to intersect at points C_{i0} and C_{i0}' of an arbitrary straight line drawn parallel to the ith line of stroke. The

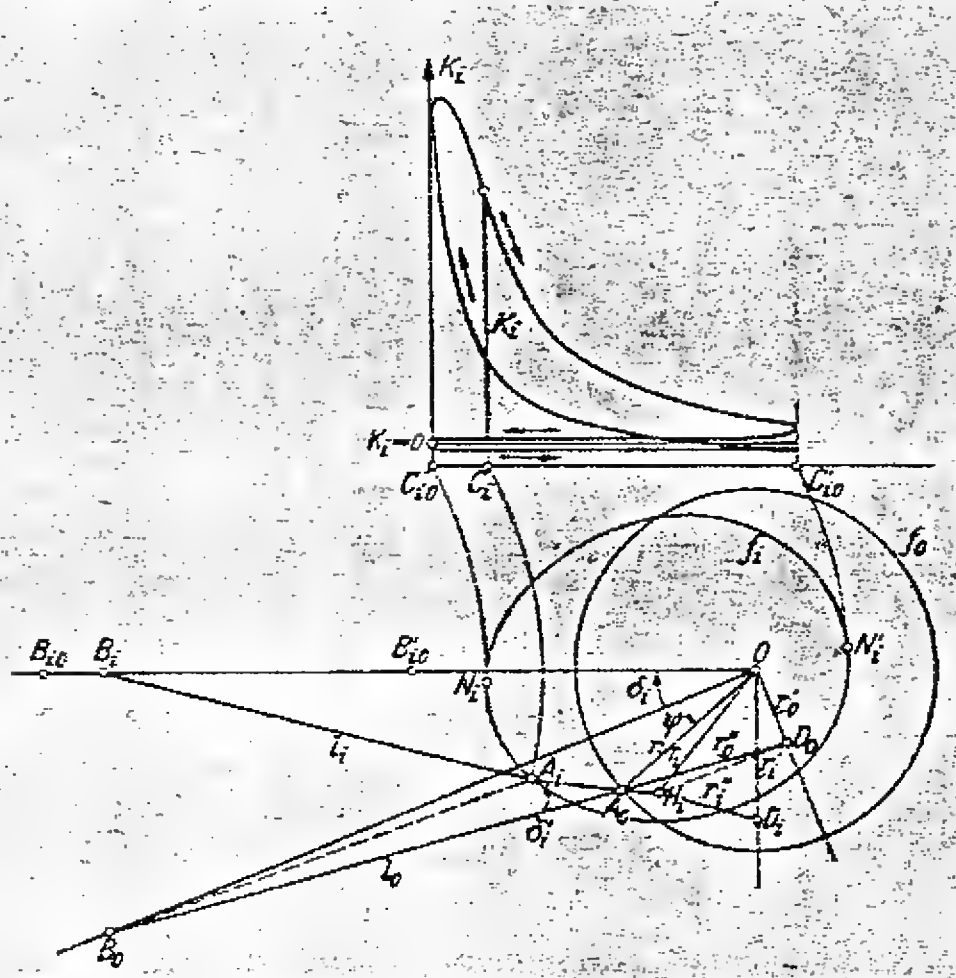


Fig. 63.—Graphical determination of the tangential crank force P_i and the corresponding torque $\mathfrak{M}_i'$ from the piston force K_i.

known indicator diagram showing the piston force K_i is placed so that the line $C_{i0}C_{i0}'$ corresponds to the stroke. Lastly, we find values of K_i and r_i' for a particular crank rotation ψ (measured from the master line of stroke). This can be done without reference to the instantaneous centres, which are sometimes rather inconvenient to use, by the method illustrated in Fig. 63: the circular arc A_iC_i about B_i determines K_i; if B_0A_0 is cut by a line $\psi = \pi/2$ at D_0 the line D_0H_i (H_i being a point on A_0A_i) can be drawn parallel to B_0A_i; finally, H_iD_i is parallel to B_iA_i and OD_i gives the desired length r_i'. From K_i and r_i' we get $\mathfrak{M}_i'$ or P_i by means of (3) or (4) and we can plot the curves giving $\mathfrak{M}_i'$ or P_i as functions of ψ. Fig. 64 shows these curves for the master mechanism and the four articulated mechanisms of the radial engine of Fig. 47 (I. 21) as well as the curve for the tangential force P or

tangential torque M′ of the whole engine as obtained by superposition.

For locating the two dead-centres the following should be considered: if the master rod were of infinite length the curve f_i would be a circle of the same size as f_0, obtained by displacing the centre O along the line $\psi = -\delta_i'$ by the distance A_0A_i. For a master rod of finite length the curve f_i is very different from the circle f_0; however, it can be shown that if δ_i and δ_i' are not very different the displaced circle and the actual curve nearly coincide in the neighbourhood of the tangent points N_i and N_i', so that C_{i0} and C_{i0}', the base points of the indicator diagram, are placed fairly accurately by using the displaced crank circle instead of the curve f_i.

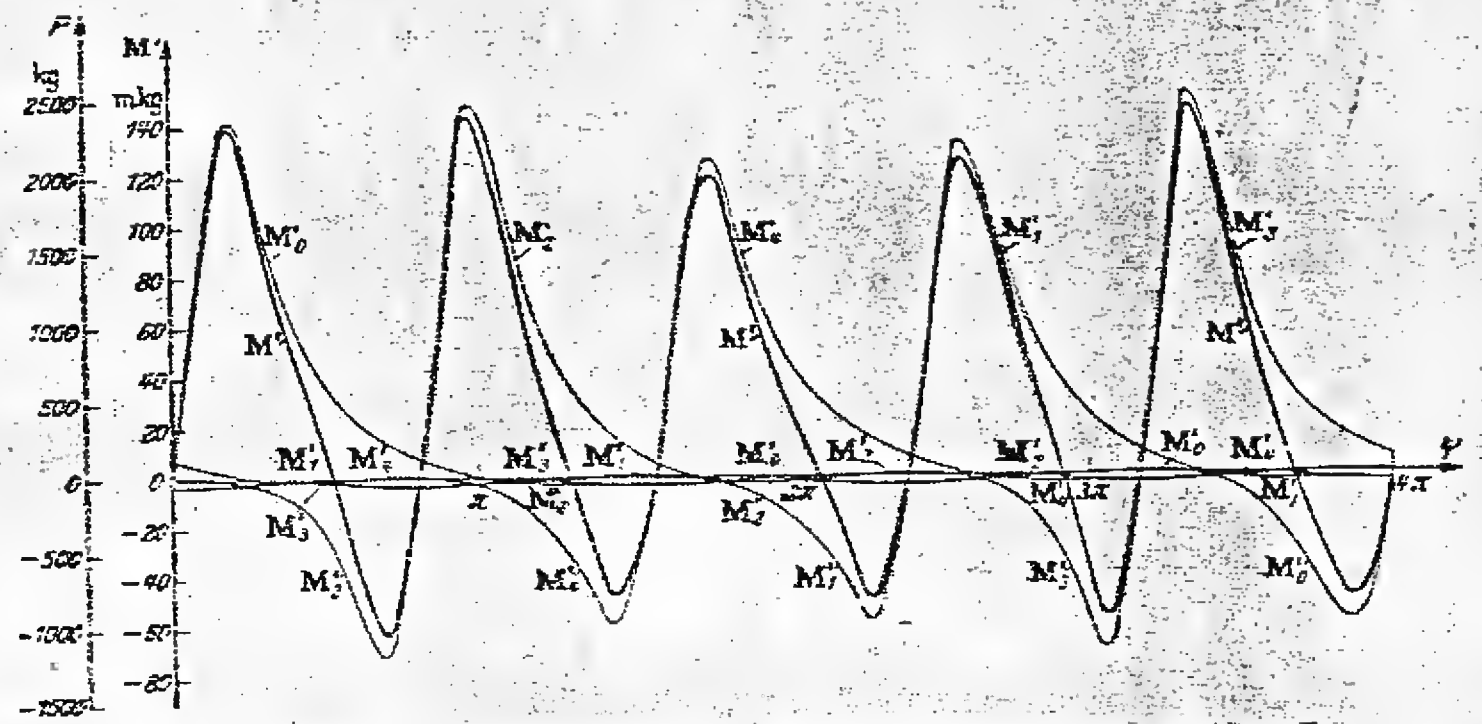


Fig. 64.—Tangential crank-force diagram for the radial engine of Fig. 47 of Part I.

To evaluate the second term M″ of the power equation we must find the kinetic energy associated with one bank of cylinders. This consists of the kinetic energy T_0 of the master mechanism and that of the satellite mechanisms T_i. The magnitude of T_0 was found in Sect. 4; it is only necessary to add the subscript 0 to all of the appropriate equations (4, 1)–(4, 3), except in the crank terms:

$$T_0 = \frac{1}{2}\Psi_0\omega^2 \quad \text{with} \quad \Psi_0 = B_0 + B_0'\left(\frac{r_0}{r}\right)^2 - B_0''\left(\frac{r_0}{r}\right)^2, \tag{5}$$

where

$$\left.\begin{aligned} B_0 &= \frac{1}{g}\left[k^2G + r^2\left(1 - \frac{s_0}{l_0}\right)G_0'\right], \\ B_0' &= \frac{r^2}{g}\left(\frac{s_0}{l_0}G_0' - G_0''\right), \\ B_0'' &= \frac{\lambda_0^2}{g}\left[s_0'l_0 - s_0' - k_0^2\right]G_0' \quad \text{with} \quad \lambda_0 = \frac{r}{l_0}. \end{aligned}\right\} \tag{6}$$

The lengths r_0' and r_0'' that are necessary here are shown in Fig. 63; G_0' includes the weight of the disc to which the articulated rods are attached.

For the kinetic energy T_i of an articulated mechanism we have

$$T_i = \frac{1}{2g}\left[\left(1-\frac{s_i'}{l_i}\right)G_i' w_i^2 + \left(\frac{s_i'}{l_i}G_i' + G_i''\right)v_i^2 + (k_{1i}^2 G_{1i} + k_{2i}^2 G_{2i})\dot{\chi}_i^2\right].$$

The symbols are defined in Fig. 62. Here k_{1i} and k_{2i} are the radii of gyration of the two weights G_{1i} and G_{2i} at A_i and B_i which are equivalent to the articulated rod. They are related by the expression

$$k_{1i}^2 G_{1i} + k_{2i}^2 G_{2i} = [k_i'^2 - s_i'(l_i - s_i')]G_i'.$$

Next we introduce the expressions

$$\left.\begin{aligned} R_i &= \frac{r^2}{g}\left(1-\frac{s_i'}{l_i}\right)G_i', \\ R_i' &= \frac{r^2}{g}\left(\frac{s_i'}{l_i}G_i' + G_i''\right), \\ R_i'' &= \frac{\lambda_i^2}{g}[s_i'(l_i - s_i') - k_i'^2]G_i' \quad \text{with } \lambda_i = \frac{r}{l_i} \end{aligned}\right\} \tag{7}$$

and get for the kinetic energy of the ith articulated connecting-rod

$$T_i = \frac{1}{2}\Psi_i \omega^2 \quad \text{with} \quad \Psi_i = R_i\left(\frac{w_i}{r\omega}\right)^2 + R_i'\left(\frac{v_i}{r\omega}\right)^2 - R_i''\left(\frac{\dot{\chi}_i}{\lambda_i\omega}\right)^2. \tag{8}$$

Using the values of $(w_i/r\omega)^2$ and $(v_i/r\omega)^2$ from (1) and (2) and the relations

$$\frac{\dot{\chi}_i}{\lambda_i\omega} = \frac{l_i\dot{\chi}_i}{r\omega} = \frac{l_i\dot{x}_i}{w_i}\cdot\frac{w_i}{r\omega} = \frac{l_i}{M_iA_i}\cdot\frac{r_i}{r} = \frac{r_i''}{r_i}\cdot\frac{r_i}{r} = \frac{r_i''}{r}$$

which are obtained from Fig. 62 by inspection, we get finally

$$\Psi_i = R_i\left(\frac{r_i}{r}\right)^2 + R_i'\left(\frac{r_i'}{r}\right)^2 - R_i''\left(\frac{r_i''}{r}\right)^2 \qquad (i = 1, 2, \ldots, p-1). \tag{9}$$

The ratios that appear in this equation can all be obtained from Fig. 63. Fig. 65 shows the Ψ_0-curve and the Ψ_i-curve for the radial engine of Fig. 47 (I, 21), together with the Ψ-curve for the whole engine obtained by superposition. For comparison, there is also given the Ψ_1^*-curve obtained on the assumption that all connecting-rods are alike and are attached to the crankpin A_0 (the weight of the connecting disc being equally divided among them); in addition, there is shown the Ψ^*-curve for the whole engine obtained by superposition of the Ψ_1^*-curve and the identical curves of the other four cylinders. The Ψ-curve gives the true effective moment of inertia of the whole engine while the Ψ^*-curve gives the effective moment of inertia with all connecting-rods alike and attached to the crank.

The two curves agree closely in their average ordinates, but the approximation Ψ^* (which is nearly linear) greatly underestimates the variation in Ψ; since it is the variation in Ψ that is important for the torque M'' (4, 7), in more accurate investigations we use the Ψ-curve and consider the eccentricity of the connecting-rod attachment.

Since the graphical determination of Ψ demands a great deal of drawing-board work we shall also, as in Sect. 4, develop a method of computation by formula. However, in order to avoid expressions that are too com-

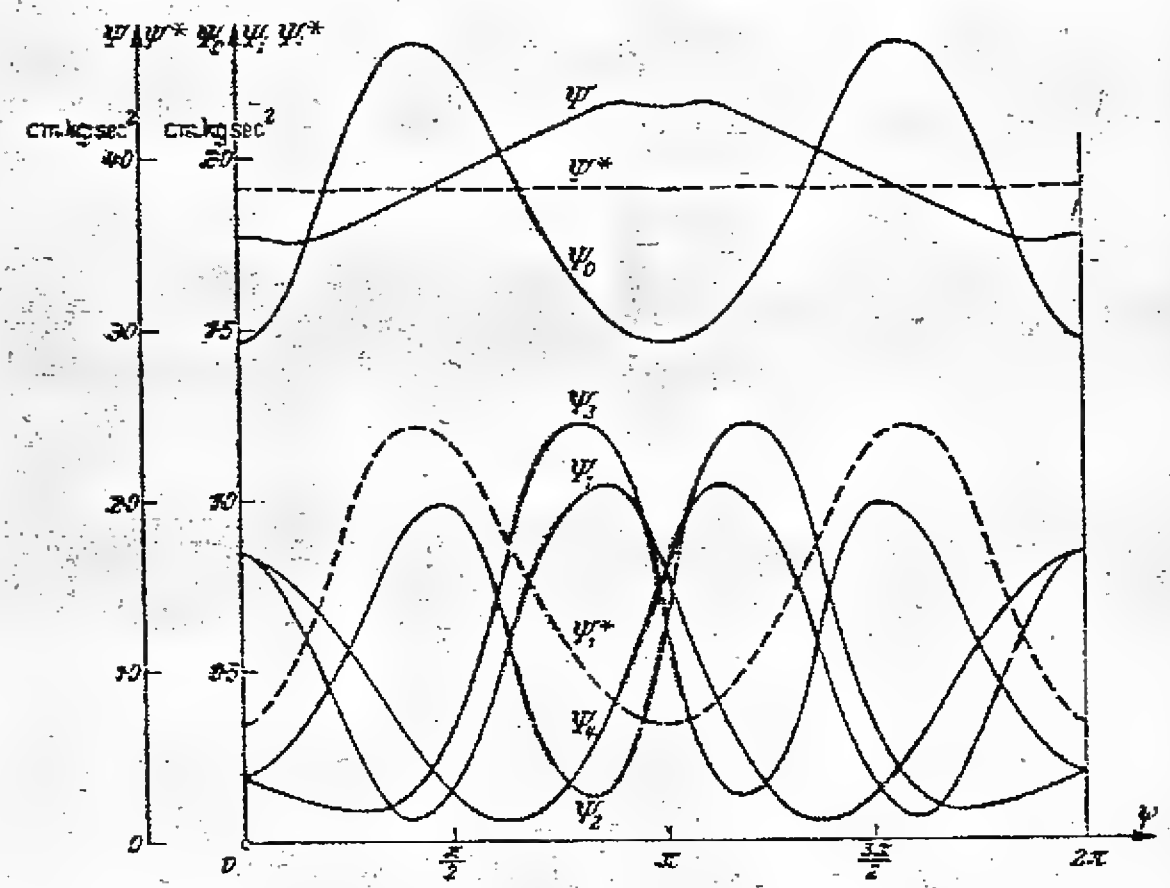


Fig. 65.—The equivalent moment of inertia Ψ of the radial engine of Fig. 47, Part I, with centrally attached (non-articulated) connecting-rods, together with the approximation Ψ^*.

plicated without losing too much accuracy we will confine ourselves to terms up to the first order in λ_0 and λ_1 and will assume $\delta_i' = \delta_i$. First, we have for the master mechanism according to (4, 2)

$$\Psi_0 = R_0 + E_0'\left(\frac{v_0}{r\omega}\right)^2 - R_0''\left(\frac{\dot{\chi}_0}{\lambda_0\omega}\right)^2. \tag{10}$$

Since $B_2 = \lambda_0/2$ and $D_1 = 1$ with consistent accuracy, we get from (4, 5) and (4, 6)

$$\left.\begin{aligned}\left(\frac{v_0}{r\omega}\right)^2 &= \frac{1}{2}(1 + \lambda_0\cos\psi - \cos 2\psi - \lambda_0\cos 3\psi),\\ \left(\frac{\dot{\chi}_0}{\lambda_0\omega}\right)^2 &= \frac{1}{2}(1 + \cos 2\psi).\end{aligned}\right\} \tag{11}$$

Likewise, for the ith mechanism,

$$\Psi_i = K_i\left(\frac{w_i}{r\omega}\right)^2 + K_i'\left(\frac{v_i}{r\omega}\right)^2 - K_i''\left(\frac{\dot{\lambda}_i}{\lambda_i\omega}\right)^2 \qquad (i = 1, 2, \ldots, p-1). \tag{12}$$

In the symbols of (I. 21) $w_i^2 = \dot{x}_i'^2 + \dot{y}_i'^2$ and $v_i = \dot{x}_i$, so that from (I, 21, 6)–(I, 21, 9) and (I, 21, 12), with some intermediate computation and with $\varepsilon_i = e_i/r$, we find

$$\left.\begin{aligned}
\left(\frac{w_i}{r\omega}\right)^2 &= 1 - \varepsilon_i\lambda_0\cos\delta_i - \varepsilon_i\lambda_0\cos\delta_i\cos 2\psi + \varepsilon_i\lambda_0\sin\delta_i\sin 2\psi,\\
\left(\frac{v_i}{r\omega}\right)^2 &= \frac{1}{2}(1 + \lambda_i\cos\delta_i\cos\psi - \cos 2\delta_i\cos 2\psi - \lambda_3\cos 3\delta_i\cos 3\psi -\\
&\qquad - \lambda_i\sin\delta_i\sin\psi + \sin 2\delta_i\sin 2\psi + \lambda_i\sin 3\delta_i\sin 3\psi),\\
\left(\frac{\dot{\lambda}_i}{\lambda_i\omega}\right)^2 &= \frac{1}{2} - \varepsilon_i\lambda_0\cos\delta_i + \left(\frac{1}{2}\cos 2\delta_i - \varepsilon_i\lambda_0\cos\delta_i\right)\cos 2\psi -\\
&\qquad - \left(\frac{1}{2}\sin 2\delta_i - \varepsilon_i\lambda_0\sin\delta_i\right)\sin 2\psi.
\end{aligned}\right\} \tag{13}$$

Finally, we have to sum these expressions and so obtain the total effective moment of inertia associated with one crank:

$$\Psi = \Psi_0 + \sum_{i=1}^{p-1}\Psi_i. \tag{14}$$

Here, p is again the number of cylinders working from the crank.

In the special case of the symmetric (not articulated) radial engine, where

$$\delta_i = i\frac{2\pi}{p} \qquad (i = 1, 2, \ldots, p-1),$$

the summations are easily made. Exactly as in (I, 24), if it is assumed that $p > 2$, we find

$$\sum_{i=1}^{p-1}\cos\delta_i = -1, \quad \sum_{i=1}^{p-1}\cos 2\delta_i = -1, \quad \sum_{i=1}^{p-1}\cos 3\delta_i = \begin{cases} 2 \\ -1 \end{cases} \quad \text{for} \quad p\begin{cases} =3 \\ >3, \end{cases}$$

$$\sum_{i=1}^{p-1}\sin\delta_i = 0, \quad \sum_{i=1}^{p-1}\sin 2\delta_i = 0, \quad \sum_{i=1}^{p-1}\sin 3\delta_i = 0,$$

and hence

$$\left.\begin{aligned}
\sum_{i=1}^{p-1}\left(\frac{w_i}{r\omega}\right)^2 &= p - 1 + \varepsilon_i\lambda_i + \varepsilon_i\lambda_0\cos 2\psi,\\
\sum_{i=1}^{p-1}\left(\frac{v_i}{r\omega}\right)^2 &= \frac{1}{2}\left(p - 1 - \lambda_i\cos\psi + \cos 2\psi + \begin{cases} -2 \\ 1 \end{cases}\lambda_3\cos 3\psi\right) \quad \text{for} \quad p\begin{cases} =3 \\ >3, \end{cases}\\
\sum_{i=1}^{p-1}\left(\frac{\dot{\lambda}_i}{\lambda_i\omega}\right)^2 &= \frac{p-1}{2} + \varepsilon_i\lambda_i - \left(\frac{1}{2} - \varepsilon_i\lambda_i\right)\cos 2\psi.
\end{aligned}\right\} \tag{15}$$

Next, to find the torque $\mathbf{M}'''$ corresponding to the potential energy $\mathbf{U}$, we consider Fig. 66, where a broken line indicates the initial state ($\psi = 0$) for which $\mathbf{U} = 0$. If we assign to point A_i the coordinates x_i', y_i', we find from this figure

$$U = -\left\{\left[sG + r\left(1 - \frac{s_0'}{l_0}\right)G_0'\right](1 - \cos\psi) + \left(\frac{s_0'}{l_0}G_0' + G_0''\right)(r + l_0 - x_0) + \right.$$
$$+ \sum_{i=1}^{p-1}\left(1 - \frac{s_i'}{l_i}\right)G_i'[(r\cos\delta_i + e_i - x_i')\cos\delta_i + (r\sin\delta_i - y_i')\sin\delta_i] +$$
$$\left. + \sum_{i=1}^{p-1}\left(\frac{s_i'}{l_i}G_i' + G_i''\right)(r\cos\delta_i + e_i + l_i\cos\chi_{0i} - x_i)\cos\delta_i\right\}.$$

The line of stroke of the master piston is taken vertically upward. We differentiate with respect to ψ to get $\mathbf{M}'''$ and introduce values for $\dot{x}_0$, $\dot{x}_i$,

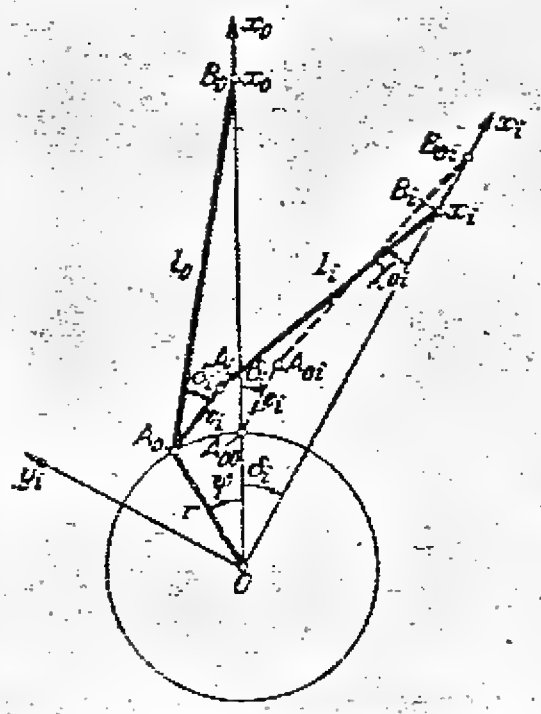


Fig. 66.—Master connecting-rod and the ith articulated rod.

$\dot{x}_i'$, $\dot{y}_i'$ from (I, 4, 5), (I, 21, 7) and (I, 21, 9). Next, we use (with subscript 0) the reduced first moments given by (5, 1) and the reduced moments for the articulated rods (assumed all identical),

$$Q_i^* = r\left(\frac{s_i'}{l_i}G_i' + G_i''\right), \qquad Q_i^{*\prime} = r\left(1 - \frac{s_i'}{l_i}\right)G_i' \tag{16}$$

finally getting, to first-order terms in λ_0 and λ_i,

$$\left.\begin{aligned}\mathbf{M}''' = -\left\{\left[Q_0^* + Q_0^{*\prime} + Q_1^*\sum_{i=1}^{p-1}\cos^2\delta_i + (p-1)Q_1^{*\prime}\right]\sin\psi + \right.\\ + \frac{1}{2}\left(\lambda_0 Q_0^* + \lambda_1 Q_1^*\sum_{i=1}^{p-1}\cos\delta_i\cos 2\delta_i\right)\sin 2\psi + \\ \left. + \left(Q_1^*\sum_{i=1}^{p-1}\sin\delta_i\cos\delta_i + \varepsilon_1\lambda_0 Q_1^{*\prime}\sum_{i=1}^{p-1}\sin\delta_i\right)\cos\psi + \frac{1}{2}\lambda_1 Q_1^*\sum_{i=1}^{p-1}\cos\delta_i\sin 2\delta_i\cos 2\psi\right\}.\end{aligned}\right\} \tag{17}$$

In the special case of the symmetrical radial engine (without articulated rods) the summations have been evaluated in (I, 24), giving the simple result

$$\left.\begin{aligned} M''' = -\Big\{\Big[Q_0^* + Q_0^{*\prime} + \Big(\frac{p}{2} - 1\Big)Q_1^* + (p-1)Q_1^{*\prime}\Big]\sin\psi + \\ + \frac{1}{2}\Big[\lambda_0 Q_0^* + \begin{pmatrix} +1 \\ -1 \end{pmatrix}\lambda_1 Q_1^*\Big]\sin 2\psi\Big\} \quad \text{for} \quad p\begin{pmatrix} =3 \\ >3 \end{pmatrix} \end{aligned}\right\} \tag{18}$$

or with reference to (I, 24, 3),

$$M''' = -\frac{g}{\omega^2}\int_0^\psi X^*\,d\psi, \tag{19}$$

where X^* symbolizes the steady longitudinal force (as follows immediately from (5, 9)). It is easily shown that formula (19) also holds for the W-engine (I, 23), but not for the V-engine, since in it the master mechanism is not normally vertical. We will not write down the explicit formulæ for the V-engine; they offer no difficulty.

In calculating the internal forces of these engines—the cylinder-wall force, the gudgeon-pin reactions and the connecting-rod force—we may take over equations (6, 1)–(6, 4) directly, the only difference being that we write $\delta = 0$ for the master mechanism, and $\delta = \delta_i$ and $\psi + \delta_i$ instead of ψ for the ith mechanism. From (I, 21, 7) and (I, 21, 12) we find that up to the first-order terms in λ_0 and λ_i, the equations (6, 9) and (6, 10) for the articulated mechanisms are exactly the same as for the master mechanism, if again $\psi + \delta_i$ replaces ψ. Moreover, as in Sect. 6, we conclude that for moderate crank/connecting-rod ratios λ_0, λ_i and eccentricities e_i, the transverse connecting-rod forces K_4 always remain negligible: $K_{40} = 0$, $K_{4i} = 0$ $(i = 1, 2, \ldots, p-1)$. Under such circumstances it is certain that the final equations (6, 15)–(6, 17) for gudgeon-pin forces K_1 and K_2, cylinder-wall forces K'' and connecting-rod forces K_3 (with the changes noted) can be taken over. Here and later we will omit the terms in λ_0 and λ_i when they are accompanied by terms without λ_0 and λ_i and are not multiplied by the (generally) very large piston force K or by the factor $r\omega^2/g$. Likewise, for simplicity we will eliminate the gravity force terms since they are never of importance. Then, for the master mechanism,

$$\left.\begin{aligned} K_{10} &= K_0 - G_0''\frac{r\omega^2}{g}\cos\psi, \qquad K_{20} = -\frac{1}{2}\lambda_0\frac{s_0'}{l_0}G_0'\frac{r\omega^2}{g}\sin 2\psi, \\ K_0'' &= \lambda_0\Big[K_4\sin\psi - \frac{1}{2}\Big(\frac{s_0'}{l_0}G_0' + G_0''\Big)\frac{r\omega^2}{g}\sin 2\psi\Big], \\ K_{30} &= K_0 - \Big(\frac{s_0'}{l_0}G_0' + G_0''\Big)\frac{r\omega^2}{g}\cos\psi. \end{aligned}\right\} \tag{20}$$

and for the other mechanisms

$$\left.\begin{aligned} K_{1i} &= K_i - G_i''\frac{r\omega^2}{g}\cos(\psi+\delta_i), \quad K_{2i} = -\frac{1}{2}\lambda_i\frac{s_i'}{l_i}G_i'\frac{r\omega^2}{g}\sin 2(\psi+\delta_i), \\ K_i'' &= \lambda_i\left[K_i\sin(\psi+\delta_i) - \frac{1}{2}\left(\frac{s_i'}{l_i}G_i' + G_i''\right)\frac{r\omega^2}{g}\sin 2(\psi+\delta_i)\right], \\ K_{3i} &= K_i - \left(\frac{s_i'}{l_i}G_i' + G_i''\right)\frac{r\omega^2}{g}\cos(\psi+\delta_i). \end{aligned}\right\} (i=1,2,\ldots,p-1) \quad (21)$$

From this point on the calculation follows a somewhat different path. On the equivalent mass G_{1i}, thought of as referring to the ith connecting-rod on the bearing A_i (Fig. 67), there act the connecting-rod force K_{3i} and

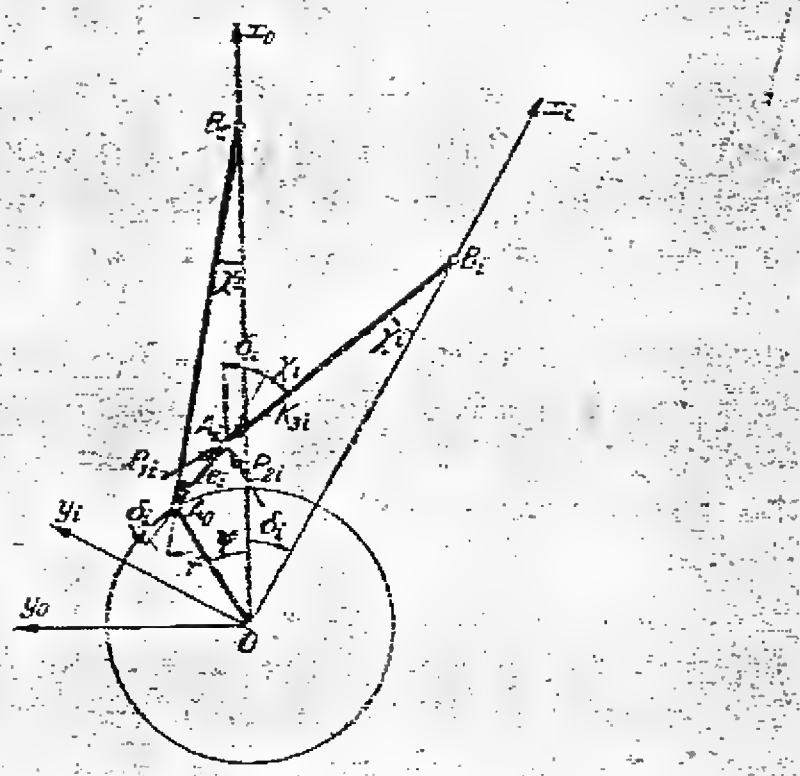


Fig. 67.—Forces acting on the equivalent mass G_{1i} of the ith articulated connecting-rod.

the bearing force, which is best handled in two components P_{1i} and P_{2i} with directions $\psi - \pi/2$ and ψ. The bearing A_i has coordinates in the (x_i, y_i)-system (I, 21, 9) as follows:

$$x_i' = r[\cos(\psi+\delta_i) + \varepsilon_i], \qquad y_i' = r\sin(\psi+\delta_i) \quad \text{with} \quad \varepsilon_i = \frac{e_i}{r}$$

in which all terms in λ_0 are omitted. In the (x_0, y_0)-system of the master mechanism the coordinates are

$$\begin{aligned} \xi_i' &= \quad x_i'\cos\delta_i + y_i'\sin\delta_i = r(\cos\psi + \varepsilon_i\cos\delta_i), \\ \eta_i' &= -x_i'\sin\delta_i + y_i'\cos\delta_i = r(\sin\psi - \varepsilon_i\sin\delta_i). \end{aligned}$$

Hence, the accelerations are

$$\ddot{\xi}_i' = -r\omega^2\cos\psi, \qquad \ddot{\eta}_i' = -r\omega^2\sin\psi.$$

The equations of motion of the equivalent mass $G_{1i} \equiv (1 - s_i'/l_i)G_i'$ are then

$$P_{1i}\sin\psi + P_{2i}\cos\psi - K_{3i}\cos(\chi_i + \delta_i) = -\left(1 - \frac{s_i'}{l_i}\right)G_i'\frac{r\omega^2}{g}\cos\psi,$$

$$-P_{1i}\cos\psi + P_{2i}\sin\psi + K_{3i}\sin(\chi_i + \delta_i) = -\left(1 - \frac{s_i'}{l_i}\right)G_i'\frac{r\omega^2}{g}\sin\psi.$$

In view of the other approximations being made, we place $\chi_i = 0$ and, on introducing K_{3i} from (21), we get the components of the forces on the bearing A_i,

$$\left.\begin{aligned} P_{1i} &= K_i\sin(\psi + \delta_i) - \left(\frac{s_i'}{l_i}G_i' + G_i''\right)\frac{r\omega^2}{2g}\sin 2(\psi + \delta_i), \\ P_{2i} &= K_i\cos(\psi + \delta_i) - \left(1 - \frac{s_i'}{l_i}\right)G_i'\frac{r\omega^2}{g} - \left(\frac{s_i'}{l_i}G_i' + G_i''\right)\frac{r\omega^2}{2g}[1 + \cos 2(\psi + \delta_i)]. \end{aligned}\right\} \quad (22)$$

In the same way we find the equations of motion of the equivalent mass G_{10}, thought of as replacing the master connecting-rod at A_0, on which act all reactions $-P_{1i}$ and $-P_{2i}$:

$$P_{10}\sin\psi + P_{20}\cos\psi - K_{30}\cos\chi - \sum_{i=1}^{p-1}(P_{1i}\sin\psi + P_{2i}\cos\psi) = -\left(1 - \frac{s_0'}{l_0}\right)G_0'\frac{r\omega^2}{g}\cos\psi,$$

$$-P_{10}\cos\psi + P_{20}\sin\psi + K_{30}\sin\chi + \sum_{i=1}^{p-1}(P_{1i}\cos\psi - P_{2i}\sin\psi) = -\left(1 - \frac{s_0'}{l_0}\right)G_0'\frac{r\omega^2}{g}\sin\psi.$$

With the values of K_{3i} from (20), P_{1i} and P_{2i} from (22) and with $\chi = 0$, the components of the bearing force at A_0 are found to be

$$\left.\begin{aligned} P_{10} &= K_0\sin\psi - \left(\frac{s_0'}{l_0}G_0' + G_0''\right)\frac{r\omega^2}{2g}\sin 2\psi + \sum_{i=1}^{p-1}\left[K_i\sin(\psi + \delta_i) - \left(\frac{s_i'}{l_i}G_i' + G_i''\right)\frac{r\omega^2}{2g}\sin 2(\psi + \delta_i)\right], \\ P_{20} &= K_0\cos\psi - \left(1 - \frac{s_0'}{l_0}\right)G_0'\frac{r\omega^2}{g} - \left(\frac{s_0'}{l_0}G_0' + G_0''\right)\frac{r\omega^2}{2g}(1 + \cos 2\psi) + \sum_{i=1}^{p-1}\left\{K_i\cos(\psi + \delta_i) - \right. \\ &\qquad \left. - \left(1 - \frac{s_i'}{l_i}\right)G_i'\frac{r\omega^2}{g} - \left(\frac{s_i'}{l_i}G_i' + G_i''\right)\frac{r\omega^2}{2g}[1 + \cos 2(\psi + \delta_i)]\right\}. \end{aligned}\right\} \quad (23)$$

In equations (6, 19) for the main bearing forces nothing is changed—of course we now use P_{10} and P_{20} which we have just found instead of P_1 and P_2.

In the special case of the regular radial engine the summations in (23) can almost always be evaluated explicitly; then for $p > 2$ and $G_0'' = G_i''$ (all pistons of equal weight) we get the following simple formulæ for the connecting-rod forces:

$$\left.\begin{aligned} P_{10} &= K_0\sin\psi + \sum_{i=1}^{p-1}K_i\sin(\psi + \delta_i) - \left(\frac{s_0'}{l_0}G_0' - \frac{s_1'}{l_1}G_1'\right)\frac{r\omega^2}{2g}\sin 2\psi, \\ P_{20} &= K_0\cos\psi + \sum_{i=1}^{p-1}K_i\cos(\psi + \delta_i) - \left(\frac{s_0'}{l_0}G_0' - \frac{s_1'}{l_1}G_1'\right)\frac{r\omega^2}{2g}\cos 2\psi - \\ &\qquad - \left\{\left[\left(1 - \frac{1}{2}\frac{s_0'}{l_0}\right)G_0' + \frac{1}{2}G_0''\right] + (p-1)\left[\left(1 - \frac{1}{2}\frac{s_1'}{l_1}\right)G_1' + \frac{1}{2}G_1''\right]\right\}\frac{r\omega^2}{g}. \end{aligned}\right\} \quad (24)$$

If we compare this result with (15, 7) for the radial engine with centrally attached connecting-rods (in which the gravity force terms must be omitted), we discover that within the error of our approximations the effect of the eccentricity of attachment of the connecting-rods appears only in the terms with the factor $(s_0' G_0'/l_0 - s_1' G_1'/l_1)$. Because the master and the articulated connecting-rods are frequently of very different shapes this effect can be important, so that high-speed engines would experience a large additional rotating force,

$$Z_1^* = \left(\frac{s_0'}{l_0} G_0' - \frac{s_1'}{l_1} G_1'\right) \frac{r\omega^2}{g}. \tag{25}$$

having an angular velocity 2ω with respect to the rotating crank and ω with respect to the fixed base of the engine.

The extension of all these results to multi-bank V-, W- and radial engines would proceed as in Sect. 13.

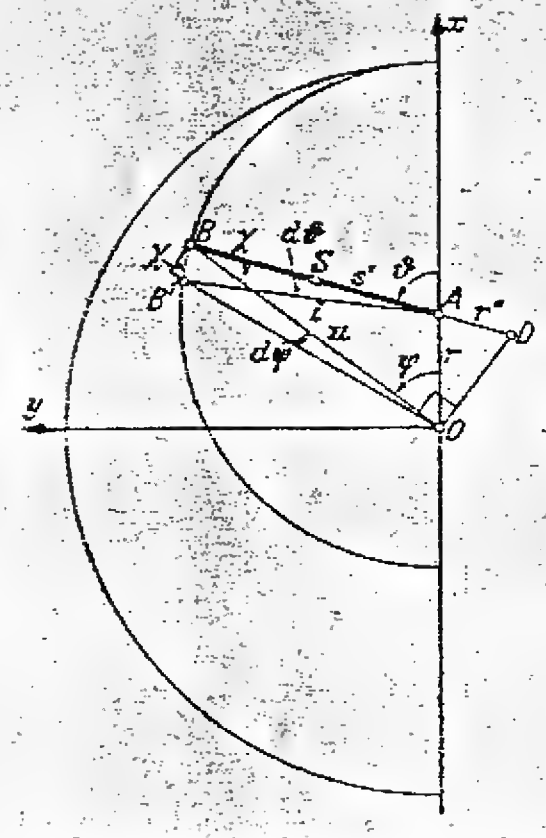


Fig. 68.—Rotary engine in two consecutive states ψ and $\psi + d\psi$.

17. The Rotary Engine.—Since the derivation of the moment $\mathfrak{M}'$ of the tangential force K (Sect. 3) is quite independent of whether the cylinders are fixed and the crank rotates or vice versa, we find for the rotary engine (I, 28) the same tangential-force moment $\mathfrak{M}'$ as for the radial engine (15 and 16).

The kinetic energy of a single mechanism of the rotary engine, in the symbols of (I, 28) and Fig. 54, or the adjacent Fig. 68, is

$$T_1 = \frac{1}{2_2} \{[(k'^2 + s'^2) G' + l^2 G''] \dot{\vartheta}^2 + k''^2 G'' \omega^2\} + \frac{1}{2} D e \dot{e}^2.$$

From Fig. 68, where OD is perpendicular to OE, we have

$$l\,d\vartheta = \frac{u\,d\psi}{\cos\chi} = (l + r'')\,d\psi$$

or

$$\vartheta = \frac{l + r''}{l}\,\omega, \tag{1}$$

so that the kinetic energy of one mechanism is

$$T_i = \frac{1}{2}\,\Psi_i\omega^2 \quad \text{with} \quad \Psi_i = D + \frac{1}{g}\left[(k^{*2}G' + l^2G'')\left(\frac{l + r''}{l}\right)^2 + k''^2G''\right]. \tag{2}$$

In this expression

$$k^{*2} = k'^2 + s'^2, \tag{3}$$

i.e. the square of the radius of gyration of the connecting-rod with respect to the centre of the crankpin A. In the effective moment of inertia Ψ_i, the moment of inertia D of the cylinder (together with the various engine elements rotating with it) with respect to O, the crank axis, will predominate while that of the piston with respect to its own centre of gravity k''^2G''/g will be relatively unimportant. The only factor in (2) that varies with ψ is $[(l + r'')/l]^2$, which can be obtained directly from Fig. 68 so that the graphical calculation of the Ψ-curve, first for one cylinder and then by superposition for the whole rotary engine, is very simple here. In the actual computation we keep the cylinder fixed and allow the crank to turn.

However, the kinetic energy (2) can also be easily evaluated mathematically. Since $\vartheta = \psi + \chi$ or $\dot\vartheta = \omega + \dot\chi$, with the value of $\dot\chi$ from (I, 4, 9), it follows from (1) that

$$\frac{l + r''}{l} = 1 + \lambda(D_1\cos\psi - D_2\cos 3\psi + D_3\cos 5\psi - + \cdots),$$

or squaring,

$$\left(\frac{l + r''}{l}\right)^2 = D_0'' + D_1''\cos\psi + D_2''\cos 2\psi - D_3''\cos 3\psi - D_4''\cos 4\psi + + - - \cdots, \tag{4}$$

in which

$$\left.\begin{aligned} D_0'' &= 1 + \lambda^2 D_0', & D_1'' &= 2\lambda D_1', \\ D_2'' &= \lambda^2 D_2', & D_3'' &= 2\lambda D_3', \\ D_4'' &= \lambda^2 D_4', & D_5'' &= 2\lambda D_5', \\ &\vdots & &\vdots \end{aligned}\right\} \tag{5}$$

Here the D_i are the Fourier coefficients tabulated in (I, 4) and the D_i' those written down in (4, 6).

For a regular rotary engine with p cylinders, summation yields for the total kinetic energy

$$T = \frac{1}{2}\Psi\omega^2 \quad \text{with}$$
$$\Psi = p\left\{D + \frac{1}{g}\left[(k^{*2}G' + l^2G'')(D_0'' \pm D_p'' \cos p\varphi \pm D_{2p}'' \cos 2p\varphi \pm \cdots) + k''^2G''\right]\right\}. \tag{6}$$

All other terms disappear. In practice, it is always sufficiently accurate to eliminate the terms containing D_p'', D_{2p}'', etc., and from (I, 4, 10) and (4, 6) to set $D_0'' = 1 + \lambda^2/2$ so that Ψ assumes the constant approximate value

$$\Psi = p\left\{D + \frac{1}{g}\left[\left(1 + \frac{1}{2}\lambda^2\right)(k^{*2}G' + l^2G'') + k''^2G''\right]\right\}. \tag{7}$$

The potential-energy torque becomes, as we can easily show,

$$\mathfrak{M}'' = -\frac{g}{\omega^2}\int_0^{\varphi} X^*\, d\varphi, \tag{8}$$

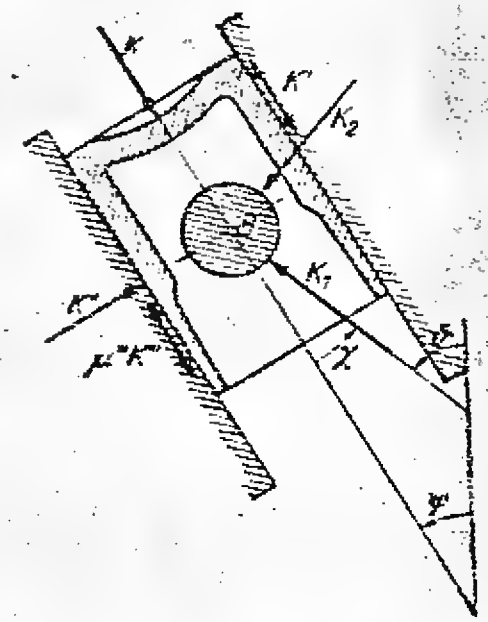


Fig. 69.—The piston and the forces acting on it.

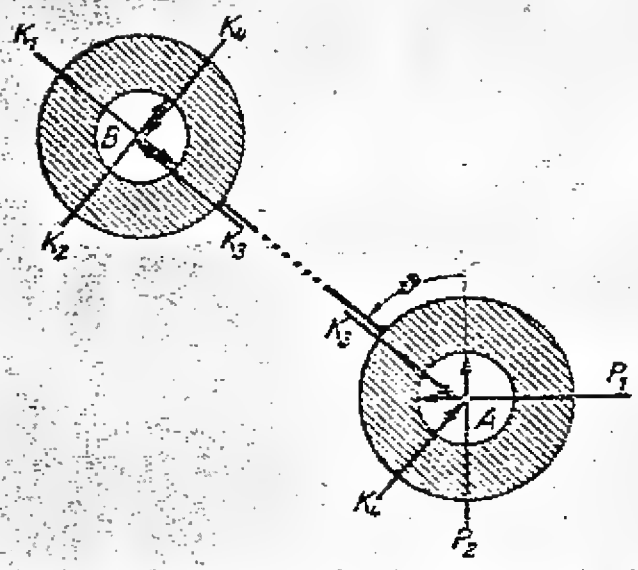


Fig. 70.—Equivalent mass system of connecting-rod and the forces on it.

provided the fixed crank is directed vertically upward. Here, X^* symbolizes the steady component of the "longitudinal" force as usual (I, 28, 4). This torque is normally very small.

From Figs. 69 and 70 the internal forces can be calculated in the same way as in Sect. 6. We shall not write down explicitly the basic relations which are of the same form as (6, 1)–(6, 6) but give only the final values, omitting all λ-terms for simplicity, and assuming very small crank radii r so that the angular rotations φ and ϑ of cylinder and connecting-rod are interchangeable. Only in the cylinder-wall force K'', which is actually the

turning force of the engine, will we keep the first-order λ-terms. With this order of approximation we find

$$\left.\begin{aligned} &K_1 = K - G''\frac{l\omega^2}{g}, \quad K_2 = 0, \quad K_3 = K - \left(\frac{s'}{l}G' + G''\right)\frac{l\omega^2}{g}, \quad K_4 = 0, \\ &K'' = \lambda\left(K + G''\frac{l\omega^2}{g}\right)\sin\psi. \end{aligned}\right\} \tag{9}$$

For small values of r, i.e. for nearly coincident cylinder and connecting-rod axes, we can verify these results immediately by remembering that the piston force K is approximately equal to the sum of the gudgeon-pin force K_1 and the centrifugal force of the piston $G''l\omega^2/g$, and also approximately equal to the sum of the connecting-rod force K_3 and the centrifugal force $(G_2 + G'')l\omega^2/g$ of the piston and the equivalent mass G_2 (of the connecting-rod on the gudgeon-pin). Finally, the cylinder-wall force K'' must be in equilibrium with the component $K_1 \sin\chi$ on one hand and on the other with the Coriolis force $2G''\omega v/g$, in which the relative velocity v of the piston with respect to the cylinder is seen immediately from Fig. 68 to be $v = l\omega\sin\chi$. Then $K'' = (K_1 + 2G''l\omega^2/g)\sin\chi$; this, with K_1 and with $\sin\chi = \lambda\sin\psi$, gives precisely the last equation (9).

The components of the crankpin force are finally found to be

$$P_1 = K_3\sin\vartheta \approx K_3\sin\psi, \qquad P_2 = K_3\cos\vartheta \approx K_3\cos\psi, \tag{10}$$

in which the connecting-rod force (see Sect. 6) is obtained from (9). With the present degree of accuracy the total crankpin force vanishes when summed for all p cylinders. Of course, the friction torques of the individual connecting-rods do not vanish when summed; they are computed individually from (10) and added.

18. The Counter-rotating Engine.—In the counter-rotating engine as well (I, 29), the moment of the tangential force $\mathfrak{M}'$ can be found by the method of Sect. 3; it is only necessary to use the relative rotation of crank and cylinders, in the symbols of (I, 29) the relative angular velocity $\omega + \omega'$, where ω is the angular velocity of the crank and $\omega' = \varkappa\omega$ is that of the cylinders. If $\omega_0 = \varkappa_0\omega$ is the actual angular velocity of the driving shaft the corresponding moment of the tangential force is

$$\mathfrak{M}_0' = \frac{\omega + \omega'}{\omega_0}\mathfrak{M}' = \frac{1 + \varkappa}{\varkappa_0}\mathfrak{M}'. \tag{1}$$

Usually, the gearing that connects crank, driving shaft and cylinders is such as to make $\varkappa = \varkappa_0 = 1$—this is assumed to be the case in the following—so that

$$\mathfrak{M}_0' = 2\mathfrak{M}'. \tag{2}$$

Moreover, the firing of a four-stroke engine is repeated with each revolution of the driving shaft, so that in comparison with the $\mathfrak{M}'$-curve for stationary

cylinders the ordinates must be doubled and the abscissæ halved; this can obviously be done without redrawing the curve, simply by changing scales.

The kinetic energy of a single mechanism, with the usual symbols and according to Fig. 71, is

$$T_i = \frac{1}{2g}\left\{\left[k^2 G + r^2\left(1-\frac{s'}{l}\right)G'\right]\omega^2 + \left(\frac{s'}{l}G' + G''\right)u^2 + (k_1^2 G_1 + k_2^2 G_2)\dot{\theta}^2 + \right.$$
$$\left. + k''^2 G''\omega^2\right\} + \frac{1}{2}D\omega^2,$$

or

$$T_i = \frac{1}{2}\Psi_i\omega^2 \quad \text{with} \quad \Psi_i = D + \frac{1}{g}\left\{\left[k^2 G + r^2\left(1-\frac{s'}{l}\right)G' + k''^2 G''\right] + \right.$$
$$\left. + r^2\left(\frac{s'}{l}G' + G''\right)\left(\frac{u}{r\omega}\right)^2 - [s'(l-s') - k''^2]\left(\frac{\dot{\theta}}{\omega}\right)^2\right\}. \qquad (3)$$

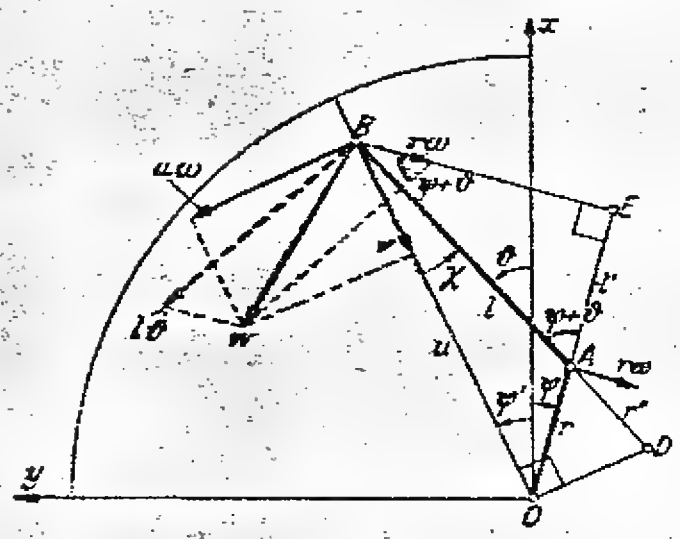


Fig. 71.—Crank and connecting-rod of counter-rotating engine.

Here, only the variable factors $(u:r\omega)^2$ and $(\dot{\vartheta}/\omega)^2$ remain to be determined. But $\vartheta = \psi' + \chi$ and, consequently (since $\dot{\psi}' = \omega$),

$$\frac{\dot{\vartheta}}{\omega} = 1 + 2\frac{\dot{\chi}}{2\omega}.$$

As in Sect. 4 (since 2ω is the relative angular velocity between crank and cylinders),

$$\frac{l\dot{\chi}}{2r\omega} = \frac{r''}{r} \quad \text{or} \quad \frac{\dot{\chi}}{2\omega} = \frac{r''}{l},$$

and hence

$$\frac{\dot{\vartheta}}{\omega} = 1 + 2\frac{r''}{l} (\equiv q), \qquad (4)$$

a quantity that can be taken directly from Fig. 71 for any crank rotation. Moreover, the velocity w of the gudgeon-pin, instead of being found from v and $u\omega$, can equally well be obtained from the velocity $r\omega$ of the crankpin

A and the relative velocity $l\dot{\vartheta}$ of the gudgeon-pin B relative to the crankpin A. Then from Fig. 71

$$w^2 = r^2\omega^2 + l^2\dot{\vartheta}^2 - 2r\omega l\dot{\vartheta}\cos(\psi + \vartheta),$$

or, using (4),

$$\left(\frac{w}{r\omega}\right)^2 = 1 + \frac{l^2}{r^2}q^2 - 2\frac{l}{r}q\cos(\psi + \vartheta),$$

or with $l' = l\cos(\psi + \vartheta) = AE$, where BE is perpendicular to AE,

$$\left(\frac{w}{r\omega}\right)^2 = 1 + \left(\frac{l}{r}\right)^2 q^2 - 2\frac{l'}{r}q. \tag{5}$$

Therefore, we need only to determine the two lengths r'' and l' for each crank position in $\psi + \psi'$ (holding the cylinders fast) and then with equations (3)–(5) to calculate the effective moment of inertia Ψ_i as function of $\psi + \psi'$. Since $\psi' = \psi + \delta$, where δ is the dead-centre angle of the cylinder from the vertical x-axis, we can next get Ψ_i as a function of $2\psi + \delta$ and therefore of ψ itself. We shall not consider the mathematical representation of T_i, which is rather awkward in this case.

The torque $\mathfrak{M}'''$ corresponding to the potential energy of the counter-rotating engine is given by the relation

$$\mathfrak{M}''' = -\frac{g}{\omega^2}\int_0^{\psi} X^*\,d\psi, \tag{6}$$

where X^* is obtained from (I, 29, 7), if it is assumed that at least one line of stroke of the regular counter-rotating engine is vertically upward in its dead-centre position (coinciding with the x-axis).

The internal forces are found as in the rotary engine. So long as we neglect terms containing λ, i.e. assume r to be very small, the quantities K_1, K_2, K_3, K_4 are exactly the same as in the rotary engine. Therefore, as in (17, 9),

$$\left.\begin{aligned} &K_1 = K - G''\frac{l\omega^2}{g}, \quad K_2 = 0, \quad K_3 = K - \left(\frac{s'}{l}G' + G''\right)\frac{l\omega^2}{g}, \quad K_4 = 0, \\ &K'' = \lambda\left(K + 3G''\frac{l\omega^2}{g}\right)\sin(2\psi + \delta). \end{aligned}\right\} \tag{7}$$

The cylinder-wall force K'', however, is quite different from that of the rotary engine; it must be in equilibrium both with the force component $K_1\sin\chi$ and with the Coriolis force $2G''\omega v/g$. But from Fig. 71 with $\psi' = \psi + \delta$

$$\sin\chi = \lambda\sin(2\psi + \delta)$$

and

$$v = l\dot{\vartheta}\sin\chi + r\omega\sin(\psi + \vartheta - \chi)$$

or since $\vartheta = \psi + \chi + \delta$,

$$v = \lambda l(\dot{\vartheta} + \omega)\sin(2\psi + \delta).$$

But since $\dot{\vartheta} = \omega$ (neglecting λ-terms),

$$v = 2\lambda l\omega \sin(2\varphi + \delta),$$

from which equation (7) for the cylinder-wall force K'' follows directly.

Finally, the components P_1 and P_2 of the crankpin reaction are

$$P_1 = K_3 \sin(2\varphi + \delta), \qquad P_2 = K_3 \cos(2\varphi + \delta), \tag{8}$$

if, as always before, P_1 is assumed tangent to the crank circle and positive when acting opposite to the direction of motion, while P_2 is positive along the crank from the axis O outward to the crankpin. The final remark of Sect. 17 applies here.

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